Critical Point Theory Applied to Bundles

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Abstract

This study was motivated by the observation that most smooth bundles do not admit a smooth function that is Morse when restricted to every fibre.

The complexity c of a critical point of a smooth map is measured by an appropriate codimension of its germ. The subset of smooth maps from a bundle to a manifold with complexity on fibres not exceeding c is studied. Bounds for c are established such that this subset is open and dense in the set of all smooth maps, where sets of smooth maps are always given the Whitney C^∞ topology. The bounds are calculated in terms of the dimensions of the base space, the fibre and the manifold into which the bundle is mapped and are proved using the theory of finite germs and a suitable adaptation of the Thom Transversality Theorem. Recent work of Vasil'ev is used to investigate real-valued functions on compact principal S^1-bundles. The existence is established of a function with complexity on fibres no more than roughly half of the minimum value for c for the open and dense subsets mentioned above.

For certain bundles with fibre of dimension one, the set of smooth real-valued functions that are Morse when restricted to every fibre is shown to be C^0 dense but not, in general, C^1 dense. For all n-sphere bundles over the circle the set is shown to be C^0 dense.

The homotopy type of the space of smooth Morse functions on the circle is derived. Arnol'd's determination of the fundamental group of the generalized Morse functions on the circle is included.

A description of a practical method for determining the index of nondegenerate critical points in Lagrange multiplier problems concludes the study. Here the signature and nullity of the bordered Hessian are related to those of the restricted Hessian.

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Chapter 1

Introduction

"Singularity is almost invariably a clue." The Adventures of Sherlock Holmes: The Boscombe Valley Mystery, Sir Arthur Conan Doyle

This thesis contains a variety of results concerning the singularities of smooth maps on bundles. We say that a smooth map on the total space of a smooth bundle obeys a property <u>on fibres</u> if its restriction to each fibre obeys that property. This study was motivated by the observation that not all smooth bundles admit a smooth real-valued function that is Morse on fibres. It is natural to ask if all smooth bundles do admit a smooth real-valued function that is a generalized Morse function on fibres. A generalized Morse function is a smooth function that is only allowed to have critical points that are nondegenerate or of the simplest degenerate kind (see §4.5 and §4.6). Here the complexity c of a critical point of a smooth map is measured by an appropriate codimension of its germ (see 3.2.14).

In Chapter 4 we study the subset of smooth maps from a bundle to a manifold with complexity on fibres not exceeding c. Bounds for c are established such that this subset is open and dense in the set of all smooth maps, where sets of smooth maps are always given the Whitney C^∞ topology. The bounds depend solely upon the dimensions of the base space, the fibre and the manifold into which the bundle is mapped and are proved using the theory of finite germs and a suitable adaptation of the Thom Transversality Theorem. One corollary is that if the dimension of the base space is zero or one, then the set of functions that are generalized Morse on fibres is dense and open.

In the second approach in Chapter 9, recent work of Vasil'ev [47] is used to investigate real-valued functions on compact principal S^1-bundles. The existence is established of a function with complexity on fibres no more than roughly half of the minimum value for c for the open and dense subsets found in Chapter 4.

In Chapter 5 we relate the results in Chapter 4 to those in the literature.

In Chapter 6 we study the set of smooth real-valued functions Morse on fibres. We prove, for certain bundles with fibre of dimension one, that this set is C^0 dense but show that, in general, it is not C^1 dense. For all n-sphere bundles over the circle the set is shown to be C^0 dense.

In Chapter 7 we determine the homotopy type of the space of smooth Morse functions on the circle. Arnol'd's determination of the fundamental group of the generalized Morse functions on the circle follows in Chapter 8.

Chapters 2 and 3 are preparations for the proofs in Chapters 4 and 5. Chapter 3 contains the theory of finite germs required for these chapters. Additional background information and notation is provided in the Appendix.

Chapter 10 is independent of the rest of the thesis. It describes a practical method for determining the index of nondegenerate critical points in Lagrange multiplier problems. The signature and nullity of the bordered Hessian are related to those of the restricted Hessian. This result has been accepted for publication [20].

Chapter 2

Preliminaries

This thesis presupposes knowledge of standard theory concerning topology, differentiable manifolds, germs, jets, fibre bundles, homotopy, Thom transversality, the Whitney topologies, group theory, Lie groups, Lie algebras and some basic algebraic geometry. Some of these notions are recalled in the Appendix, especially where it is convenient to have notation for the body of the thesis. A comprehensive treatment of these notions may be found in [3, 6, 13, 15, 22, 42, 44, 49].

In the remainder of this chapter some of these notions are modified to deal with the restriction of maps on the total space of a bundle to each fibre.

2.1 Parametrized Morse Functions

Let $\pi : E \to B$ be a smooth fibre bundle where the fibre is a manifold of dimension n and the base B is a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. For every b in B, denote the fibre $\pi^{-1}(b)$ by F_b.

Let Y be a smooth p-manifold, $1 \leq p < \infty$. The restriction

$$f_b \;:\; F_b \;\to\; Y$$

of any smooth map $f : E \to Y$ is then smooth for each b in B.

Definition 2.1.1 *Following [17], we define the critical graph of the smooth map f to be*

$$\Gamma_f = \{e \in E : f_{\pi(e)} \text{ has a critical point at } e\}.$$

The tangent bundle along the fibres of π is defined to be

$$T_\pi E = Ker(d\pi : TE \to TB).$$

We recall the definition of a Morse function. Let X be a smooth n-manifold and let

$$g : X \to \mathbf{R}$$

be a smooth function. A point x in X is a nondegenerate critical point of g if and only if x is a critical point of g and the Hessian form of g at x is nonsingular. It is proved in [35, page 4] that, at a critical point x of g, the Hessian form of g is nonsingular if and only if, in any choice of local co-ordinates about x in X, the matrix of second partial derivatives of g is nonsingular. The index of a nondegenerate critical point of g is the maximal dimension of a subspace of T_xX, the tangent space to X at x, on which the Hessian form of g at x is negative definite.

The map

$$g : X \to \mathbf{R}$$

is Morse if every critical point of g is nondegenerate.

It is proved in [35, page 8] that nondegenerate critical points are isolated. Hence, if X is compact and $g : X \to \mathbf{R}$ is Morse, than g has a finite number of critical points.

Definition 2.1.2 *Following [17], a smooth map $f : E \to \mathbf{R}$ is called a parametrized Morse function if f is Morse on fibres.*

The following proposition and corollaries, proved in [17], describe the properties of the critical graph of a parametrized Morse function.

Proposition 2.1.3 *Let f in $C^\infty(E, \mathbf{R})$ be a parametrized Morse function.*

1. *The critical graph Γ_f forms a submanifold of E whose dimension is $\dim(B)$ and whose normal bundle is the restriction of $T_\pi E$.*

2. *If Γ_i, $1 \leq i \leq k$, denote the connected components of Γ_f and the fibre of $\pi : E \to B$ is compact then*

$$\pi : \Gamma_i \to B$$

   is a finite covering map for each i.

3. *The function*

$$Ind : \Gamma_f \to \mathbf{Z}_+ = \{0, 1, 2, 3, \ldots\}$$

   given by the index of the critical point in the corresponding fibre, is constant on each component Γ_i.

4. *The normal bundle of a component Γ_i contains a subbundle whose dimension is given by the index and so the normal bundle has a corresponding splitting.*

Corollary 2.1.4 *If B is simply connected, then the degree of the covering map $\pi : \Gamma_i \to B$ is equal to one for every i, $1 \leq i \leq k$, and so Γ_i is diffeomorphic to B.*

Corollary 2.1.5 *Not every bundle $\pi : E \to B$ admits a parametrized Morse function.*

Proof If B is simply connected and π does not admit a section then there can be no parametrized Morse function on E since Γ_i would define a section, any i, $1 \leq i \leq k$. For example, not one of the Hopf bundles (see A.4)

$$\pi^n : S^{2n+1} \to \mathbf{CP}^n, \quad n \geq 1,$$

admits a parametrized Morse function. □

2.2 An Adaptation of the Thom Transversality Theorem

An adaptation of the Thom Transversality Theorem is essential for the proofs of the main results in Chapter 4. The Thom Transversality Theorem is stated below and proved in [15, page 54]. It has been adapted for our study of smooth maps on bundles, restricted to the fibres. Our proof of the adaptation follows the proof in [15].

Definition 2.2.1 *Let X be a topological space. Then a subset $Y \subseteq X$ is residual if it is the countable intersection of open dense subsets of X [15, page 54].*

Definition 2.2.2 *A topological space X is a Baire space if every residual subset is dense [15, page 44].*

Notation 2.2.3 *Let X and Y be smooth manifolds. Then the smooth maps from X into Y are denoted $C^\infty(X, Y)$.*

Proposition 2.2.4 *Let X and Y be smooth manifolds. Then $C^\infty(X, Y)$ is a Baire space in the Whitney C^∞ topology (see [15, pages 42, 44]).*

For proof see [15, page 44].

Notation 2.2.5 *Let X and Y be smooth manifolds. Then the smooth manifold of k-jets from X into Y is denoted $J^k(X,Y)$.*

Notation 2.2.6 *Denote by*

$$j^k \ : \ C^\infty(X,Y) \ \rightarrow \ C^\infty(X, J^k(X,Y))$$

the continuous map which assigns to every element f in $C^\infty(X,Y)$ the smooth map

$$j^k f \ : \ X \ \rightarrow \ J^k(X,Y)$$

which in turn assigns to every element x in X the k-jet $j^k f(x)$ (see A.6).

Notation 2.2.7 *Let X and Y be smooth manifolds and $f : X \rightarrow Y$ a smooth map. Let W be a submanifold of Y. Then "f intersects W transversally" is denoted "$f \pitchfork W$" (see [15, page 50]).*

Theorem 2.2.8 The Thom Transversality Theorem
If X and Y are smooth manifolds, W is a submanifold of $J^k(X,Y)$, $1 \leq k < \infty$, and

$$T_W = \{f \in C^\infty(X,Y) \ : \ j^k f \pitchfork W\}$$

then T_W is a residual subset of $C^\infty(X,Y)$ in the Whitney C^∞ topology.

A similar statement is required for the smooth maps from the total space of a smooth bundle into a smooth manifold, restricted to the fibres.

Let $\pi : E \rightarrow B$ be a smooth fibre bundle with fibre a manifold X of dimension n and with base B a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Let Y be a smooth p-manifold, $1 \leq p < \infty$. Let

$$\tilde{\pi} \ : \ J^k(E,Y) \ \rightarrow \ E \times Y$$

be the smooth projection defined in A.6.

Notation 2.2.9 *Denote by* $J^{k,0}_{n,p}$ *the space of k-jets of germs* : $(\mathbf{R}^n, \underline{0}) \to (\mathbf{R}^p, \underline{0})$, $1 \leq n,\ p \leq \infty,\ 1 \leq k \leq \infty$ *(see A.6.6).*

Notation 2.2.10 *For finite k, denote by* L^k_n *the set of germs of diffeomorphisms in* $J^{k,0}_{n,n}$ *with the subspace topology.*

Lemma 2.2.11 *The smooth manifold* $J^k(E, Y)$ *is the total space of a smooth fibre bundle with base* $E \times Y$*, fibre* $J^{k,0}_{m+n,p}$*, projection* $\tilde{\pi}$ *and group* $L^k_m \times L^k_n \times L^k_p$*, where* $k,\ n,\ p \geq 1$ *and* $m \geq 0$ *(we take* $L^k_0 \times L^k_n$ *to be* L^k_n*).*

Proof Let the diffeomorphisms

$$\Psi_i \ : \ \mathbf{R}^m \ \to \ V_i, \ \ i \in \mathcal{I},$$

determine co-ordinate patches on the smooth manifold B such that there exist maps

$$\Phi_i \ : \ V_i \times X \ \to \ \pi^{-1}(V_i), \ \ i \in \mathcal{I},$$

which determine smooth local trivialisations of the bundle π. Let

$$\Psi'_j \ : \ \mathbf{R}^n \ \to \ U_j, \ \ j \in \mathcal{J},$$

and

$$\Psi''_r \ : \ \mathbf{R}^p \ \to \ W_r, \ \ r \in \mathcal{R},$$

determine co-ordinate patches on the smooth manifolds X and Y respectively. Define

$$\begin{array}{rcll} \Psi_{ij} \ : \ \mathbf{R}^m \times \mathbf{R}^n & \to & \pi^{-1}(V_i) & \text{by} \\ (\underline{b}, \underline{x}) & \mapsto & \Phi_i(\Psi_i(\underline{b}), \Psi'_j(\underline{x})) & \end{array}$$

where $\underline{b} \in \mathbf{R}^m$ and $\underline{x} \in \mathbf{R}^n$. Then

$$\{\Phi_i(V_i \times U_j) \ : \ i \in \mathcal{I},\ j \in \mathcal{J}\}$$

and

$$\{\Psi_{ij} \; : \; i \in \mathcal{I}, \, j \in \mathcal{J}\}$$

is a system of smooth co-ordinates in E.

A canonical map

$$\Phi_{ijr} \;\; : \;\; \Phi_i(V_i \times U_j) \times W_r \times J^{k,0}_{m+n,p} \;\; \rightarrow \;\; (\tilde{\pi}^{-1})(\Phi_i(V_i \times U_j) \times W_r)$$

is defined by mapping

$$(\Phi_i(\Psi_i(\underline{b}), \Psi'_j(\underline{x})), \Psi''_r(\underline{y}), g)$$

to the jet in

$$(\tilde{\pi}^{-1})(\Phi_i(\Psi_i(\underline{b}), \Psi'_j(\underline{x})), \Psi''_r(\underline{y}))$$

which in local co-ordinates is

$$f \;\; : \;\; (\mathbf{R}^m \times \mathbf{R}^n, (\underline{b}, \underline{x})) \;\; \rightarrow \;\; (\mathbf{R}^p, \underline{y})$$

where

$$f(\underline{b}', \underline{x}') = g(\underline{b}' - \underline{b}, \underline{x}' - \underline{x}) + \underline{y},$$

where $i \in \mathcal{I}$, $j \in \mathcal{J}$, $r \in \mathcal{R}$, $\underline{b}' \in \mathbf{R}^m$, $\underline{b} \in \mathbf{R}^m$, $\underline{x}' \in \mathbf{R}^n$, $\underline{x} \in \mathbf{R}^n$, $\underline{y} \in \mathbf{R}^p$ and $g \in J^{k,0}_{m+n,p}$.

For (e, y) in $\Phi_i(V_i \times U_j) \times W_r$, let $\Phi_{ijr,ey}$ be the restriction of Φ_{ijr} to $(e, y) \times J^{k,0}_{m+n,p}$. Then

$$\Phi_{ijr,ey} \;\; : \;\; J^{k,0}_{m+n,p} \;\; \rightarrow \;\; (\tilde{\pi}^{-1})(e, y)$$

is a diffeomorphism for every such choice of i, j, r, e and y. Hence, for every choice of i, i', j, j', r, r', e and y such that
$(e, y) \in (\Phi_i(V_i \times U_j) \times W_r) \cap (\Phi_{i'}(V_{i'} \times U_{j'}) \times W_{r'})$ the map

$$\Phi^{-1}_{i'j'r',ey} \circ \Phi_{ijr,ey} \;\; : \;\; J^{k,0}_{m+n,p} \;\; \rightarrow \;\; J^{k,0}_{m+n,p}$$

is an element of $L_m^k \times L_n^k \times L_p^k$. Equipped with the product topology, the space $L_m^k \times L_n^k \times L_p^k$ is an effective topological transformation group of $J_{m+n,p}^{k,0}$. Clearly, the map $\Psi_{i'j'r'ijr}$ from $(\Phi_i(V_i \times U_j) \times W_r) \cap (\Phi_{i'}(V_{i'} \times U_{j'}) \times W_{r'})$ into $L_m^k \times L_n^k \times L_p^k$ defined by

$$(e, y) \mapsto \Phi_{i'j'r',ey}^{-1} \circ \Phi_{ijr,ey}$$

is smooth for every choice of i, i', j, j', r and r'. Hence

$$\{\Phi_i(V_i \times U_j) \times W_r \,:\, i \in \mathcal{I},\, j \in \mathcal{J},\, r \in \mathcal{R}\}$$

and

$$\{\Psi_{i'j'r'ijr} \,:\, i,\, i' \in \mathcal{I},\, j,\, j' \in \mathcal{J},\, r,\, r' \in \mathcal{R}\}$$

is a system of smooth co-ordinate transformations in $E \times Y$.

We have established the data for a bundle with base space $E \times Y$, fibre $J_{m+n,p}^{k,0}$, group $L_m^k \times L_n^k \times L_p^k$ and the co-ordinate transformations $\{\Psi_{i'j'r'ijr} \,:\, i,\, i' \in \mathcal{I},\, j,\, j' \in \mathcal{J},\, r,\, r' \in \mathcal{R}\}$ (see [44, Existence Theorem, page 14]). □

Next, we construct a smooth fibre bundle whose total space is equal to the union of the spaces $J^k(F_b, Y)$, b in B. Let $\tilde{\Psi}_{i'j'r'ijr}$ be the restriction of $\Psi_{i'j'r'ijr}$ to $L_n^k \times L_p^k$ for every i, i' in $\mathcal{I}$, j, j' in $\mathcal{J}$, r and r' in $\mathcal{R}$. Similarly to the above proof, there exists a smooth bundle with base space $E \times Y$, fibre $J_{n,p}^{k,0}$, group $L_n^k \times L_p^k$, the system of smooth co-ordinate transformations

$$\{\Phi_i(V_i \times U_j) \times W_r \,:\, i \in \mathcal{I},\, j \in \mathcal{J},\, r \in \mathcal{R}\}$$

and

$$\{\tilde{\Psi}_{i'j'r'ijr} \,:\, i,\, i' \in \mathcal{I},\, j,\, j' \in \mathcal{J},\, r,\, r' \in \mathcal{R}\}$$

and total space equal to

$$\{\sigma \in J^k(F_b, Y) \,:\, b \in B\}.$$

We shall call the total space of this smooth fibre bundle $J_{fibre}^k(E, Y)$.

Definition 2.2.12 *For $k \geq 1$ denote by*

$$j^k_{fibre} \; : \; C^\infty(E,Y) \;\to\; C^\infty(E, J^k_{fibre}(E,Y))$$

the continuous map which assigns to every element f in $C^\infty(E,Y)$ the smooth map

$$\begin{array}{rcll} j^k_{fibre}f & : & E \;\to\; J^k_{fibre}(E,Y) & \text{defined by} \\ & & e \;\mapsto\; j^k f_{\pi(e)}. & \end{array}$$

The map $j^k_{fibre}f$ defined above is smooth as it is a restriction of the smooth map $j^k f \; : \; E \to J^k(E,Y)$.

Theorem 2.2.13 An Adaptation of the Thom Transversality Theorem for Fibre Bundles *Let $\pi \; : \; E \to B$ be a smooth fibre bundle with fibre a manifold of dimension n and base B of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Let Y be a smooth p-manifold, $1 \leq p < \infty$ and W a submanifold of $J^k_{fibre}(E,Y)$, for some k, $1 \leq k < \infty$. If*

$$T_W = \{f \in C^\infty(E,Y) \; : \; j^k_{fibre} f \pitchfork W\}$$

then T_W is a residual subset of $C^\infty(E,Y)$ in the Whitney C^∞ topology.

Proof We need to show that T_W is the countable intersection of open dense subsets. To construct the sets which will be intersected we choose a countable covering of W by open sets $W_1, W_2, \ldots$ such that each W_r satisfies

1. the closure of W_r in $J^k_{fibre}(E,Y)$ is contained in W,

2. $\overline{W}_r$ is compact,

3. there exist co-ordinate neighbourhoods U_r in E and V_r in Y such that $\tilde{\pi}_{fibre}(\overline{W}_r) \subset U_r \times V_r$ where

   $$\tilde{\pi}_{fibre} \; : \; J^k_{fibre}(E,Y) \;\to\; E \times Y$$

   is the bundle projection and

4. $\overline{U}_r$ and $\overline{V}_r$ are compact.

This choice is possible since W is a submanifold of $J^k_{fibre}(E,Y)$ and so around each point w in W we may choose an open set W_w satisfying the four requirements above. Since W is second countable we may extract a countable subcovering from $\{W_w\}_{w\in W}$. Let

$$T_{W_r} = \{f \in C^\infty(E,Y) \,:\, j^k_{fibre} f \pitchfork W \text{ on } \overline{W}_r\}.$$

It is clear that

$$T_W = \cap_{r=1}^{\infty} T_{W_r}.$$

Thus the proof reduces to showing that each T_{W_r} is open and dense in $C^\infty(E,Y)$.

Define

$$T_r = \{g \in C^\infty(E, J^k_{fibre}(E,Y)) \,:\, g \pitchfork W \text{ on } \overline{W}_r\}.$$

By Proposition 2.2.15, T_r is open. Since

$$j^k_{fibre} \;:\; C^\infty(E,Y) \;\to\; C^\infty(E, J^k_{fibre}(E,Y))$$

is continuous it follows that

$$T_{W_r} = (j^k_{fibre})^{-1}(T_r)$$

is open.

Next we show T_W is dense. Choose charts

$$\psi \;:\; U_r \;\to\; \mathbf{R}^{n+m}, \quad U_r \subset E$$

$$\eta \;:\; V_r \;\to\; \mathbf{R}^p, \quad V_r \subset Y$$

and smooth functions

$$\rho \;:\; \mathbf{R}^{n+m} \;\to\; [0,1]$$

and

$$\rho' \ : \ \mathbf{R}^p \ \rightarrow \ [0,1]$$

such that

$$\rho = \begin{cases} 1 & \text{on a neighbourhood of } \psi \circ \alpha(\overline{W}_r) \\ 0 & \text{off } \psi(U_r) \end{cases}$$

and

$$\rho' = \begin{cases} 1 & \text{on a neighbourhood of } \eta \circ \beta(\overline{W}_r) \\ 0 & \text{off } \eta(V_r) \end{cases}$$

where $n = \dim(X)$, $m = \dim(B)$, $p = \dim(Y)$, α is the source map and β is the target map (see A.6) and such that

$$d(\text{supp}(\rho'), \mathbf{R}^p \backslash \eta(V_r))$$

is not zero where d is the usual Euclidean metric and $\text{supp}(\rho')$ is the support of ρ', that is, the closure of

$$\{\underline{x} \in \mathbf{R}^p \ : \ \rho'(\underline{x}) \neq 0\}.$$

It is possible to choose ρ and ρ' as $\overline{W}_r$ is compact.

We show that any f in $C^\infty(E, Y)$ may be perturbed slightly to be transversal to $\overline{W}_r$. Let P' be the space of polynomial mappings from $\mathbf{R}^{n+m}$ into $\mathbf{R}^p$ of degree k. For p in P' define

$$g_p \ : \ E \ \rightarrow \ Y \ \text{ by}$$

$$g_p(e) = \begin{cases} f(e) & \text{if } e \notin U_r \text{ or } f(e) \notin V_r \\ \eta^{-1}[\rho(\psi(e))\rho'(\eta(f(e)))p(\psi(e)) + \eta(f(e))] & \text{otherwise.} \end{cases}$$

The choice of ρ and ρ' guarantees that g_p is a smooth function from E into Y and is just locally a polynomial perturbation of f smoothed out so that it is equal to f off the domain of interest. Define

$$\Phi(e, p) = j^k_{fibre} g_p(e).$$

The map g_p is a perturbation of f, hence $\Phi(e, p)$ is a perturbation of $j^k_{fibre} f(e)$, given by p.

We can show that T_{W_r} is dense by applying Proposition 2.2.14 if we can show that $\Phi \pitchfork W$ on $\overline{W}_r$. It is not necessarily true that the transversality condition holds on all of $E \times P'$, but we will find P, an open neighbourhood of 0 in P', so that

$$\Phi \; : \; E \times P \;\; \rightarrow \;\; J^k_{fibre}(E, Y)$$

is transverse to W on some neighbourhood of $\overline{W}_r$. We can then apply the proposition to $E \times P$ instead of $E \times P'$.

Assuming that the transversality condition holds, given $f : E \rightarrow Y$ we can find a sequence $p_1, p_2, \ldots$ in P converging to 0 in P so that

$$j^k_{fibre} g_{p_i} \pitchfork W$$

on $\overline{W}_r$. Since $g_0 = f$ and $g_{p_i} = f$ off U_r, it follows that $\lim_{i \to \infty} g_{p_i} = f$ in $C^\infty(E, Y)$ and T_{W_r} is dense in $C^\infty(E, Y)$.

We select our neighbourhood P. Let

$$\epsilon = \frac{1}{2} \min \{ d(supp(\rho'), \mathbf{R}^p \backslash \eta(V_r)), d(\eta \beta(\overline{W}_r), {\rho'}^{-1}[0, 1)) \}$$

where d is the usual Euclidean metric. Set

$$P = \{ p \in P' \; : \; |p\psi(e)| < \epsilon, \; \forall e \in supp(\rho \circ \psi) \}.$$

Then P is an open neighbourhood of 0 in P'. Suppose that $(e, p) \in E \times P$ and that $\Phi(e, p) \in \overline{W}_r$. We show that

$$\Phi \; : \; E \times P \;\; \rightarrow \;\; J^k_{fibre}(E, Y)$$

is locally a diffeomorphism. If true, Φ would be transverse to any submanifold in $J^k_{fibre}(E, Y)$. Since $\Phi(e, p) \in \overline{W}_r$ we have that $e \in \alpha(\overline{W}_r)$ and that $g_p(e) \in \beta(\overline{W}_r)$. Then

$$s = d(\eta f(e), \eta g_p(e)) < \epsilon$$

since

$$\eta g_p(e) = \rho(\psi(e))\rho'(\eta(f(e)))p(\psi(e)) + \eta(f(e)).$$

Using the definition of ϵ we observe that

$$\eta(f(e)) \in \text{Interior}(\rho'^{-1}(1))$$

since $g_p(e) \in \beta(\overline{W}_r)$. Recall ρ is defined to be 1 on a neighbourhood of $\psi\alpha(\overline{W}_r)$ so that

$$\eta g_p(e) = p(\psi(e)) + \eta(f(e))$$

and

$$g_p(e') = \eta^{-1}(p(\psi) + \eta(f))(e')$$

$\forall e'$ in a neighbourhood of e. This argument also holds for all p' in some neighbourhood of p in P. It follows that

$$\Phi \ : \ E \times P \ \rightarrow \ J^k_{fibre}(E, Y)$$

is locally a diffeomorphism near (e, p). For let σ be in $J^k_{fibre}(E, Y)$ near $\Phi(e, p)$, let $e' = \alpha(\sigma)$ and let p' be the unique polynomial mapping of degree $\leq k$ such that

$$\sigma = j^k_{fibre}\eta^{-1}(p'(\psi) + \eta(f))(e').$$

Then $\sigma \mapsto (e', p')$ is a smooth mapping, the inverse of Φ. □

The following two propositions are required for the proof above. The first is a modified version of [15, Lemma 4.6, page 53] and the second is [15, Proposition 4.5, page 52].

Proposition 2.2.14 *Let $\pi : E \rightarrow B$ be a smooth fibre bundle with fibre X and let Y and Z be smooth manifolds, Z a submanifold of Y, Z closed in Y. Let f be an element of $C^\infty(E, Y)$, transversal to Z. Then the set*

$$B' = \{b \in B \ : \ f_b \pitchfork Z\}$$

is dense and open in B.

Proof 1. We prove B' is dense in B. Let $Z(f) = f^{-1}(Z)$. Since $f \pitchfork Z$, $Z(f)$ is a submanifold of E. Let π_Z be the restriction of π to $Z(f)$. First note that if $b \notin \mathrm{Im}(\pi_Z)$, then $f_b(F_b) \cap Z = \emptyset$ so $f_b \pitchfork Z$. If $\dim(Z_f) < \dim(B)$ then $\pi_Z(Z(f))$ has measure zero in B, so for a dense set in B, namely $B' = (B - \mathrm{Im}(\pi_Z))$, $f_b \pitchfork Z$. Thus we may assume that $\dim(Z_f) \geq \dim(B)$. We claim that if b is a regular value of π_Z, then $f_b \pitchfork Z$. If this claim is true then the proposition is proved by Sard's Theorem. (See [15, page 30].)

To prove the claim, let b be a regular value of π_Z and let e be in F_b. If $e \notin Z(f)$, then $f_b(e) \notin Z$ and $f_b \pitchfork Z$ at e. So we may assume that $e \in Z(f)$. Since b is a regular value of π_Z and $\dim(Z_f) \geq \dim(B)$ we have that

$$T_e E = T_e Z(f) + T_e(F_b).$$

Hence

$$df_e(T_e E) = T_{f_b(e)} Z + (df_b)_e T_e(F_b).$$

Now we assumed $f \pitchfork Z$ so

$$T_{f(e)} Y = T_{f(e)} Z + df_e(T_e E).$$

Combining these two equalities,

$$T_{f_b(e)} Y = T_{f_b(e)} Z + (df_b)_e T_e(F_b).$$

Thus $f_b \pitchfork Z$ at e.

2. We prove B' is open in B. Let b be in B'. Choose $U_b \subset B$, U_b an open neighbourhood of b and a fibre preserving diffeomorphism Φ such that $\pi^{-1}(U_b) = \Phi(U_b \times X)$.

Define a smooth map

$$\begin{array}{rcccl} g & : & U_b & \to & C^\infty(X, Y) \quad \text{by} \\ & & b' & \mapsto & g_{b'} \end{array}$$

where $g_{b'}(x)$ is equal to $f_{b'}(\Phi(b', x))$, for every b' in U_b and for every x in X. Note that $g_{b'} \pitchfork Z$ iff $f_{b'} \pitchfork Z$. Hence $g_b \pitchfork Z$.

By Proposition 2.2.15,

$$\{h \in C^\infty(X, Y) \,:\, h \pitchfork Z\}$$

is open so there exists an open neighbourhood G of g_b in $C^\infty(X, Y)$ such that $h \in G$ implies $h \pitchfork Z$. Let V_b equal $g^{-1}(G)$. Then $b \in V_b$ where V_b is an open neighbourhood of b in U_b and $b' \in V_b$ implies $g_{b'} \pitchfork Z$ which in turn implies $f_{b'} \pitchfork Z$. So B' is open. □

Proposition 2.2.15 *Let X and Y be smooth manifolds and let Z be a submanifold of Y. Let*

$$T_Z = \{f \in C^\infty(X, Y) \,:\, f \pitchfork Z\}.$$

Then T_Z is an open subset of $C^\infty(X, Y)$ in the Whitney C^1 topology and thus in the Whitney C^∞ topology if Z is a closed submanifold of Y.

For proof see [15, page 52].

The corollary to the following proposition is a technical result required for the proofs in Chapter 4.

Proposition 2.2.16 *Let X and Y be smooth manifolds and let Z be a closed subset of Y. Let*

$$T_Z = \{f \in C^\infty(X, Y) \,:\, f(X) \cap Z = \emptyset\}.$$

Then T_Z is an open subset of $C^\infty(X, Y)$ in the Whitney C^1 topology and thus in the Whitney C^∞ topology.

Proof The target map $\beta : J^1(X,Y) \to Y$ is smooth (see [15, Theorem 2.7, page 41]). Since Z is a closed subset of Y, the pre-image $\beta^{-1}(Z)$ is closed in $J^1(X,Y)$. Hence $J^1(X,Y)\backslash\beta^{-1}(Z)$ is open in $J^1(X,Y)$. Observe that

$$T_Z = \{f \in C^\infty(X,Y) \,:\, j^1 f(X) \subset J^1(X,Y)\backslash\beta^{-1}(Z)\}.$$

Hence T_Z is open in $C^\infty(X,Y)$ in the Whitney C^1 topology and thus in the Whitney C^∞ topology. □

Corollary 2.2.17 *Let $\pi : E \to B$ be a smooth fibre bundle with fibre X, let Y be a smooth manifold and let Z be a closed subset of $J^k_{fibre}(E,Y)$. Let*

$$T_Z = \{f \in C^\infty(E,Y) \,:\, j^k_{fibre}f(E) \cap Z = \emptyset\}.$$

Then T_Z is an open subset of $C^\infty(E,Y)$ in the Whitney C^1 topology and thus in the Whitney C^∞ topology.

Proof Let

$$S_Z = \{g \in C^\infty(E, J^k_{fibre}(E,Y)) \,:\, g(E) \cap Z = \emptyset\}.$$

By Proposition 2.2.16, S_Z is open in the Whitney C^1 topology and thus in the Whitney C^∞ topology. Since the map

$$j^k_{fibre} \;:\; C^\infty(E,Y) \;\to\; C^\infty(E, J^k_{fibre}(E,Y))$$

is continuous,

$${j^k_{fibre}}^{-1}(S_Z) = T_Z$$

is open in the Whitney C^1 topology and thus in the Whitney C^∞ topology. □

2.3 Proalgebraic Sets

Consider the following sequence of Euclidean spaces and projections:

$$J^{\infty,0}_{n,p} \xrightarrow{\pi^\infty_{k+1}} J^{k+1,0}_{n,p} \xrightarrow{\pi^{k+1}_k} J^{k,0}_{n,p} \xrightarrow{\pi^k_{k-1}} J^{k-1,0}_{n,p} \to \dots$$

The map

$$\pi_k^\infty \ : \ J_{n,p}^{\infty,0} \ \rightarrow \ J_{n,p}^{k,0}$$

is the canonical projection. It makes sense to define algebraic sets in the finite dimensional real vector spaces $J_{n,p}^{k,0}$, $k \geq 1$. The analogous concept in infinite dimensions is a set $A \subseteq J_{n,p}^{\infty,0}$,

$$A = \cap_{k=1}^{\infty} (\pi_k^\infty)^{-1} A_k$$

where A_k is an algebraic set in $J_{n,p}^{k,0}$. We call such a set A proalgebraic. Suppose $\pi_k^{k+1} A_{k+1} \subseteq A_k$, each $k \geq 1$. Define the codimension of A to be

$$\operatorname{codim}(A) = \sup_{k\geq 1}(\operatorname{codim}(A_k)).$$

If the spaces $J_{n,p}^{k,0}$ are given the Zariski topology then the proalgebraic sets are the closed sets of the weakest topology on $J_{n,p}^{\infty,0}$ such that all the projections π_k^∞ are continuous (see [6, page 116] or A.8).

Chapter 3

The Codimension of a Germ

This chapter recalls theory of finite germs as discussed in Martinet [32] and Wall [48]. The literature also contains two concepts of germs of finite codimension (see [32] and [46]), but we will not be concerned with these. The main aim of the chapter is to describe a co-ordinate independent method of assigning to each germ of a smooth map a number which measures its complexity. The measure of complexity that is useful for the proofs in the next chapter is the codimension of the contact orbit (see §3.2) of the smooth germ in an appropriate space.

We work with smooth real germs. However, the theory included in this chapter holds for holomorphic and real analytic germs without major modification.

We give no proofs. The reader is referred to the relevant pages in Martinet [32] and Wall [48] at the appropriate points.

3.1 Notation

Let n and p be strictly positive integers and let k be a strictly positive integer or infinity.

Notation 3.1.1 *1. The vector space of germs at the origin of smooth functions : $\mathbf{R}^n \to \mathbf{R}^p$ is denoted by $E_{n,p}$. We often write E_n for $E_{n,1}$ and consider it as a ring with the ring structure induced from the ring structure on the target space $\mathbf{R}$.*

2. *The vector space of germs of smooth functions : $(\mathbf{R}^n, \underline{0}) \to (\mathbf{R}^p, \underline{0})$ is denoted by $E^0_{n,p}$. The ring $E^0_{n,1}$ is often written E^0_n.*

3. *The ideal in E_n generated by the homogeneous polynomials of degree k is denoted by M^k_n for all finite k.*

4. *The E_n-submodule of $E_{n,p}$ of those germs whose composition with each of the p co-ordinate projections : $\mathbf{R}^p \to \mathbf{R}$ lies in M^k_n is denoted by $M^k_{n,p}$.*

5. *The set of germs of local diffeomorphisms in $E^0_{n,n}$ is denoted by L_n.*

3.2 Contact Equivalence

An equivalence relation is defined on the infinite dimensional vector space $E^0_{n,p}$. The equivalence classes will be called contact classes and are the orbits in $E^0_{n,p}$ with respect to the action of a group that we will call the contact group. Well-behaved elements of $E^0_{n,p}$ belong to orbits which are infinite dimensional submanifolds of $E^0_{n,p}$ with finite codimension. The complexity of a germ in $E^0_{n,p}$ will be measured by the codimension of its orbit in $E^0_{n,p}$.

Denote by $K_{n,p}$ the set of germs of diffeomorphisms ϕ in L_{n+p} such that

1. $\phi(\underline{x}, \underline{y}) = (h(\underline{x}), \psi(\underline{x}, \underline{y}))$ where $\underline{x} \in \mathbf{R}^n$, $\underline{y} \in \mathbf{R}^p$, $\psi(\underline{x}, \underline{y}) \in \mathbf{R}^p$;

2. $h \in L_n$;

3. $\psi_{\underline{0}} \in L_p$ where $\psi_{\underline{x}}(\underline{y}) = \psi(\underline{x}, \underline{y})$ and

4. $\psi(\underline{x}, \underline{0}) = \underline{0}$, for all $\underline{x}$.

Any germ in $K_{n,p}$ may be considered as a local change of co-ordinates at the origin in $\mathbf{R}^n$ and an $\underline{x}$ dependent local change of co-ordinates at the origin in $\mathbf{R}^p$. It follows that $K_{n,p}$ is a group under the operation of composition in L_{n+p} (see [32, page 207]).

Let f be in $E^0_{n,p}$ and consider the graph of f. It is an n-dimensional submanifold of $\mathbf{R}^n \times \mathbf{R}^p$, transverse at the origin to $\{\underline{0}\} \times \mathbf{R}^p$. A local diffeomorphism $\phi = (h, \psi)$ in $K_{n,p}$ maps the graph of f to the graph of the germ defined by

$$\underline{x} \mapsto \psi(h^{-1}(\underline{x}), f \circ h^{-1}(\underline{x}))$$

which is the graph of a new germ in $E^0_{n,p}$ we call $\phi.f$. Note that the set of zeroes of f is mapped by h to the set of zeroes of $\phi.f$. This means that the graphs of f and $\phi.f$ "have the same contact" with $\mathbf{R}^n \times \{\underline{0}\}$.

Definition 3.2.1 *The group $K_{n,p}$ is called the contact group of L_{n+p}.*

Lemma 3.2.2 *The map from $E^0_{n,p} \times K_{n,p}$ into $E^0_{n,p}$ defined by*

$$(f, \phi) \mapsto \phi.f$$

is an action of the contact group $K_{n,p}$ on $E^0_{n,p}$ (see [32, page 209]).

Definition 3.2.3 *The orbit $K_{n,p}.f$ of a germ f in $E^0_{n,p}$ is called the contact orbit of f.*

Definition 3.2.4 *Two germs f and g in $E^0_{n,p}$ are contact equivalent if f and g belong to the same contact orbit.*

Contact equivalence is an equivalence relation on $E^0_{n,p}$ (see [48, page 482]). For this reason, the contact orbits are also called contact classes.

Example 3.2.5 *The germ in L_{1+1} defined by*

$$\phi(x,y) = (x, xy + y)$$

is in $K_{1,1}$. If $f \in E^0_{1,1}$ then

$$\phi.f(x) = f(x)(1+x).$$

Definition 3.2.6 *For f in $E^0_{n,p}$, let f^*M_p be the ideal in E^0_n generated by the component germs $f_1, \ldots, f_p$.*

Remark 3.2.7 *Martinet [32, page 212] proves that two map germs f and g in $E^0_{n,p}$ are contact equivalent iff there exists a local diffeomorphism h in L_n such that*

$$h^*(g^*M_p) = f^*M_p$$

where $h^*(g^*M_p)$ is equal to $(g \circ h)^*M_p$.

Definition 3.2.8 *For f in $E^0_{n,p}$, denote by $f^*M_pE_{n,p}$ the E_n-submodule of $E^0_{n,p}$ whose elements, when composed with each of the p co-ordinate projections $: \mathbf{R}^p \to \mathbf{R}$, are in f^*M_p.*

Definition 3.2.9 *For f in $E^0_{n,p}$, denote by $J(f)$ the E_n-submodule of $E_{n,p}$ generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$.*

Definition 3.2.10 *For f in $E^0_{n,p}$, the subspace TKf of $E^0_{n,p}$ is defined by*

$$TKf = M_n\, J(f) + f^*M_p\, E_{n,p}\,.$$

It may be proved (see [48, page 496]) that if the codimension of TKf in $E^0_{n,p}$ is finite then $K_{n,p}.f$ is an infinite dimensional submanifold of $E^0_{n,p}$ with codimension equal to the codimension of TKf. Hence the codimension of TKf in $E^0_{n,p}$ is invariant under local changes of co-ordinates in the domain and range and is therefore a possible measure of complexity for germs of smooth maps between manifolds. The notation TKf used above is standard and refers to the tangent space of the contact orbit of f.

Definition 3.2.11 *For f in $E^0_{n,p}$, denote by codim(TKf) the codimension of TKf in $E^0_{n,p}$ as a real vector space.*

Example 3.2.12 *Let f in $E^0_{n,p}$ be a submersion. Then, up to a local diffeomorphism in the domain, f is defined by*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}.$$

Hence $J(f)$ equals $E_{n,p}$ and codim(TKf) is zero.

Example 3.2.13 *Let f in $E^0_{n,p}$ be an immersion. Then, up to a local diffeomorphism in the range, f is defined by*

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence $f^*M_pE_{n,p}$ equals $E^0_{n,p}$ and codim(TKf) is zero.

Hence if f in $E^0_{n,p}$ has a regular point at the origin then codim(TKf) is zero. We prove later (see Lemma 4.3.4) that the converse is true.

Definition 3.2.14 *The germ f in $E^0_{n,p}$ has a singularity of codimension c at the origin if the origin is a critical point of f and*

$$codim(TKf) = c.$$

If X and Y are smooth manifolds and if $f : X \to Y$ is a smooth map then we say f has a singularity of codimension c at x in X if x is a critical point of f and in some local co-ordinates about x in X and $f(x)$ in Y,

$$codim(TKf) = c.$$

3.3 Finitely Determined Germs

Definition 3.3.1 *The germ f in $E^0_{n,p}$ is of finite type if $codim(TKf)$ is finite. The germ f in $E^0_{n,p}$ is of infinite type if $codim(TKf)$ is infinite.*

Definition 3.3.2 *The germ f in $E^0_{n,p}$ is k-K determined or k-contact determined if the set of germs with the same k-jet as f is contained in the orbit $K_{n,p}.f$. (Recall k is a strictly positive integer or infinity.) We say f is finitely K determined or finitely contact determined iff f is k-K determined for some k in $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.*

Theorem 3.3.3 *1. If k is finite and*

$$M^k_{n,p} \subseteq TKf$$

then f is k-K determined.

2. If f is k-K determined for some finite k, then

$$M^{k+1}_{n,p} \subseteq TKf$$

and so $codim(TKf)$ is finite.

Theorem 3.3.3 is due to Mather (see [48, page 486]).

Corollary 3.3.4 *The germ f in $E^0_{n,p}$ is of finite type iff f is finitely K determined.*

Proof 1. If $\operatorname{codim}(TKf) \leq k$ and k is finite then

$$M^{k+1}_{n,p} \subseteq TKf$$

and so, by Theorem 3.3.3, f is $(k+1)$-K determined.

2. If f is finitely K determined then $\operatorname{codim}(TKf)$ is finite by Theorem 3.3.3. $\square$

An important series of papers by Mather [34] laid the foundations for the study of the conditions for a germ to be finitely determined.

Chapter 4

Generic Singularity Properties of Maps on Bundles

In this chapter we prove a variety of results concerning the singularities on fibres of smooth maps on bundles. We study singularity properties of maps which are generic (see [1, 7, 25]). The first two sections of this chapter are preparation.

The proof attributed to Mather and Tougeron that the germs of infinite type are rare is outlined in §4.1. In §4.2 we construct submanifolds in the space of k-jets of smooth germs from a smooth bundle to a smooth manifold. The main result is proved in §4.3. Using the adaptation of the Thom Transversality Theorem given in §2.2, the existence is established of an open dense subset of the smooth maps from a bundle to a manifold whose elements have germs of finite type everywhere on fibres. Recall that a germ of finite type is of finite codimension. Values for the codimension c are calculated such that each map in this open dense subset has a germ of codimension not exceeding c everywhere on fibres. These values depend only upon the dimensions of the base space and fibre and the target manifold. In §4.4 we study isolated critical points on fibres. Further results in terms of A_k singularities and parametrized generalized Morse functions are given in §4.5 and §4.6.

4.1 Germs of Infinite Type are Rare

The proof that the germs of infinite type are rare (see [48, page 513]) is attributed to Mather (unpublished, see [48]) and Tougeron [46].

As in the previous chapter, let n and p be strictly positive integers and let k be a strictly positive integer or infinity. It is shown that the set of germs in $J^{\infty,0}_{n,p}$ of infinite type is a proalgebraic set of infinite codimension (see §2.3). For later work it is convenient to define, for every n, p and finite k, the set $W^k_{n,p}$ by

$$W^k_{n,p} = \{g \in J^{k,0}_{n,p} \ : \ \dim J^{k,0}_{n,p} / TK^k g \geq k-1\}.$$

Here $TK^k g$ is the set of k-jets of elements in TKg. Note that $TK^k g$ depends only upon the k-jet of g.

Lemma 4.1.1 *If $f \in E^0_{n,p}$ then $j^k f \in W^k_{n,p}$ iff $\dim J^{k,0}_{n,p} / TK^k g \geq k-1$ iff $\dim E^0_{n,p} / TKf \geq k-1$ (see [6, page 115]). Here g equals $j^k f$.*

Proof If $j^k f \notin W^k_{n,p}$, then $\dim J^{k,0}_{n,p} / TK^k g \leq k-2$.
In this case, $\dim E^0_{n,p} / (TKf + M^{k+1}_{n,p}) \leq k-2$ so by Nakayama's Lemma (A.7), setting $\mathcal{R} = E_n$, $\mathcal{K} = \mathbf{R}$, $\mathcal{M} = M_n$, $C = E^0_{n,p}$, $A = TKf$ and $d = k-1$,

$$M^k_{n,p} \subseteq TKf.$$

Hence $\dim E^0_{n,p} / TKf = \dim E^0_{n,p} / (TKf + M^{k+1}_{n,p}) \leq k-2$.

Conversely, if $\dim E^0_{n,p} / TKf \leq k-2$, then $\dim E^0_{n,p} / (TKf + M^{k+1}_{n,p}) \leq k-2$ and so by Nakayama's Lemma,

$$M^k_{n,p} \subseteq TKf.$$

Hence $\dim E^0_{n,p} / TKf = \dim J^{k,0}_{n,p} / TK^k g \leq k-2$. □

We now make use of the algebraic geometry results in A.8 and §2.3. Let

$$\begin{array}{rcl} \pi_k^\infty : J_{n,p}^{\infty,0} & \to & J_{n,p}^{k,0} \quad \text{defined by} \\ f & \mapsto & j^k f \end{array}$$

and

$$\begin{array}{rcl} \pi_k^l : J_{n,p}^{l,0} & \to & J_{n,p}^{k,0} \quad \text{defined by} \\ f & \mapsto & j^k f, \end{array}$$

$l \geq k$, be the canonical projections discussed in §2.3.

Definition 4.1.2 *Let*

$$W_{n,p}^\infty = \cap_{k\geq 1} (\pi_k^\infty)^{-1} W_{n,p}^k \subset J_{n,p}^{\infty,0}.$$

The germ f in $E_{n,p}^0$ is of infinite type iff $j^k f \in W_{n,p}^k$ for every k in $\mathbf{N}$, iff $jf \in W_{n,p}^\infty$. Note that

$$W_{n,p}^\infty = \{jf \in J_{n,p}^{\infty,0} \ : \ f \in E_{n,p}^0 \text{ and } \dim J_{n,p}^{\infty,0}/TKf = \infty\}.$$

Lemma 4.1.3 *For every n and p and every k in $\mathbf{N}$,*

1. *$W_{n,p}^k$ is an algebraic set in $J_{n,p}^{k,0}$,*
2. *$W_{n,p}^{k+1} \subseteq (\pi_k^{k+1})^{-1} W_{n,p}^k$ and*
3. *$W_{n,p}^\infty$ is a proalgebraic set in $J_{n,p}^{\infty,0}$.*

Proof 1. We show $W_{n,p}^k$ is an algebraic set. Denote by $J_{n,p}^k$ the space of k-jets of germs $: (\mathbf{R}^n, \underline{0}) \to \mathbf{R}^p$. If $f \in E_{n,p}^0$, then $j^k f \in W_{n,p}^k$ iff $\dim J_{n,p}^{k,0}/TK^k f \geq (k-1)$ iff $\dim(TK^k f) \leq \dim J_{n,p}^{k,0} - (k-1) = r(k)$.

Let

$$x^I = x_1^{I_1} x_2^{I_2} \dots x_n^{I_n}$$

where $I = (I_1, \dots, I_n)$, $I_i \in \mathbf{Z}$, $I_i \geq 0$, $1 \leq i \leq n$.

Then

$$\Phi = \{x^I \ : \ \Sigma_{i=1}^n I_i \leq k\}$$

is a basis for $J_{n,1}^k$ and

$$\Phi^0 = \{x^I \ : \ 1 \leq \Sigma_{i=1}^n I_i \leq k\}$$

is a basis for $J_{n,1}^{k,0}$.

Let

$$x^{I,j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^I \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where x^I is in the j-th position, $1 \leq j \leq p$.

Then

$$\Psi = \{x^{I,j} \ : \ \Sigma_{i=1}^n I_i \leq k, \, 1 \leq j \leq p\}$$

is a basis for $J_{n,p}^k$ and

$$\Psi^0 = \{x^{I,j} \ : \ 1 \leq \Sigma_{i=1}^n I_i \leq k, \, 1 \leq j \leq p\}$$

is a basis for $J_{n,p}^{k,0}$.

Given a in $J_{n,p}^k$ we write

$$a = \Sigma_{I,j} a_{I,j} x^{I,j}$$

where $a_{I,j} \in \mathbf{R}$ and $0 \leq \Sigma_{i=1}^n I_i \leq k$, $1 \leq j \leq p$.

If $f \in E_{n,p}^0$ then $j^k f \in W_{n,p}^k$ iff the linear map

$$\begin{array}{rcrcll} \beta(j^k f) & : & (\mathbf{R}^n \otimes J_{n,1}^{k,0}) \oplus (\mathbf{R}^p \otimes J_{n,p}^k) & \rightarrow & J_{n,p}^{k,0} & \text{given by} \\ & & e_t \otimes \phi & \mapsto & j^k(\frac{\partial f}{\partial x_t}\phi) & \text{and} \\ & & h_s \otimes \psi & \mapsto & j^k(f_s \psi) & \end{array}$$

has rank $\leq r(k)$, where $\phi \in \Phi^0$ and $\psi \in \Psi$. Here $\{e_t\}_{t=1}^n$ is the standard basis for $\mathbf{R}^n$ and $\{h_s\}_{s=1}^p$ is the standard basis for $\mathbf{R}^p$. This condition is determined by the vanishing of certain determinants which are polynomials in the coefficients $a_{I,j}$. Hence $W_{n,p}^k$ is an algebraic set.

2. Assume $j^k f \notin W_{n,p}^k$. Then

$$\begin{aligned} & \dim J_{n,p}^{k,0}/TK^k f \leq k-2 \\ \Rightarrow\ & \dim E_{n,p}^0/TKf \leq k-2 \text{ (by Nakayama's Lemma)} \\ \Rightarrow\ & j^{k+1} f \notin W_{n,p}^{k+1} \\ \Rightarrow\ & W_{n,p}^{k+1} \subseteq (\pi_k^{k+1})^{-1} W_{n,p}^k. \end{aligned}$$

3. By definition, $W_{n,p}^\infty$ is a proalgebraic set (see §2.3). □

We require the following results to show that $W_{n,p}^\infty$ is a proalgebraic set of infinite codimension in $J_{n,p}^{\infty,0}$ for every n and p.

Lemma 4.1.4 *The codim$W_{n,p}^\infty = \infty$ iff for every k in $\mathbf{N}$ and every k-jet g in $J_{n,p}^{k,0}$, there exists an l-jet h in $J_{n,p}^{l,0}$ for some finite $l \geq k$, such that $\pi_k^l h = g$ and $h \notin W_{n,p}^l$, that is, over any jet there is one of finite type (see [6, page 117]).*

Proof Assume that the supremum of the codimensions of the $W_{n,p}^k$ is infinite and that no jet of finite type lies over g in $J_{n,p}^{k,0}$. Then

$$(\pi_k^l)^{-1} g \subseteq W_{n,p}^l$$

for every $l > k$ and

$$\operatorname{codim}(\pi_k^l)^{-1} g \geq \operatorname{codim} W_{n,p}^l.$$

But then, for every $l > k$,

$$\dim J_{n,p}^{k,0} \geq \operatorname{codim}(\pi_k^l)^{-1} g \geq \operatorname{codim} W_{n,p}^l.$$

This is a contradiction.

Conversely, suppose that over each k-jet in $J^{k,0}_{n,p}$ there is a jet of finite type. Let $d_k = \mathrm{codim} W^k_{n,p}$, then

$$d_k \leq d_{k+1} \leq d_{k+2} \ldots$$

as $W^{k+1}_{n,p} \subseteq (\pi^l_k)^{-1} W^k_{n,p}$. The proof is finished unless eventually

$$d_k = d_{k+1} = d_{k+2} \ldots$$

In this case, consider an irreducible component X_k of $W^k_{n,p}$ with highest dimension. By Theorem A.8.11, $(\pi^l_k)^{-1} X_k$ is irreducible for $l \geq k$, and so, for every $l \geq k$ either

1. $W^l_{n,p} \cap (\pi^l_k)^{-1} X_k = (\pi^l_k)^{-1} X_k$ or
2. $\mathrm{codim}(W^l_{n,p} \cap (\pi^l_k)^{-1} X_k) > \mathrm{codim}(\pi^l_k)^{-1} X_k$.

Let b_k equal the number of irreducible components of $W^k_{n,p}$ with highest codimension. We argue that

$$b_k \geq b_{k+1} \geq b_{k+2} \ldots$$

Any irreducible component of $W^{k+1}_{n,p}$ is contained in the lift of $W^k_{n,p}$ under $(\pi^{k+1}_k)^{-1}$ and so in the lift of irreducible component of $W^k_{n,p}$. Since $d_k = d_{k+1}$, an irreducible component of $W^k_{n,p}$ with highest dimension is contained in, and is hence equal to, $(\pi^{k+1}_k)^{-1} X_k$ where X_k is a similar component of $W^k_{n,p}$. Now b_k is finite, hence eventually

$$b_k = b_{k+1} = b_{k+2} = \ldots$$

This means we always have the first situation above, that is, no irreducible components of highest dimension are lost. It follows that

$$(\pi^l_k)^{-1} X_k \subseteq W^l_{n,p}$$

for every $l > k$. Therefore if g is in X_k, then h in $(\pi_k^l)^{-1}X_k$ will be in $W_{n,p}^l$. This is a contradiction. □

Lemma 4.1.5 *For every k in $\mathbb{N}$ and every k-jet g in $J_{n,p}^{k,0}$, there exists an l-jet h in $J_{n,p}^{l,0}$ for some finite $l \geq k$ such that $\pi_k^l h = g$ and $h \notin W_{n,p}^l$.*

A proof appears in [48, page 513]. It has been omitted due to its length. It requires the use of complex analytic germs and the Nullstellensatz for Coherent Sheaves.

Corollary 4.1.6 *The set $W_{n,p}^{\infty,0}$ is a proalgebraic set of infinite codimension in $J_{n,p}^{\infty,0}$.*

4.2 Singularity Subbundles

Let $\pi : E \to B$ be a smooth fibre bundle with fibre a manifold of dimension n and base B a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Let Y be a smooth p-manifold, $1 \leq p < \infty$, and consider the space of smooth maps $C^\infty(E,Y)$ with the Whitney C^∞ topology.

In preparation for the proofs in the following sections we construct, for each k in $\mathbb{N}$, a finite set of subbundles in $J^k_{fibre}(E,Y)$ whose union contains all the k-jets of infinite type. Recall from §2.2, $J_{n,p}^{k,0}$ is the fibre of the projection

$$\tilde{\pi}_{fibre} \;:\; J^k_{fibre}(E,Y) \;\to\; E \times Y$$

and

$$J^k_{fibre}(E,Y) = \{\sigma \in J^k(F_b, Y) \;:\; b \in B\}.$$

For fixed k in $\mathbb{N}$, the union of these subbundles is the set $W_{n,p}^k$ when projected onto any fibre. These subbundles have codimensions whose minimum increases with k as does the codimension of $W^k_{n,p}$. If k is chosen large enough, the codimension of each subbundle in $J^k_{fibre}(E,Y)$ is greater than the dimension of E.

By Theorem A.8.9, $W^k_{n,p}$ is the disjoint union of a finite number of analytic manifolds we call $W^{k,i}_{n,p}$, such that the dimension of $W^{k,i}_{n,p}$ is less than or equal to the dimension of $W^k_{n,p}$, $1 \leq i \leq i_k$. We write

$$W^k_{n,p} = \coprod_{i=1}^{i_k} W^{k,i}_{n,p}.$$

The submanifolds $W^{k,i}_{n,p}$ are unique up to order.

Lemma 4.2.1 *There exists a finite disjoint union of subbundles in $J^k_{fibre}(E,Y)$ whose projection into any fibre is the set $W^k_{n,p}$. The codimension of each subbundle is at least the codimension of $W^k_{n,p}$ in $J^{k,0}_{n,p}$.*

Proof Recall that if $f \in E^0_{n,p}$, $\phi \in L_n$ and $\psi \in L_p$, then

$$j^k f \in W^k_{n,p} \Leftrightarrow j^k(\psi \circ f \circ \phi) \in W^k_{n,p}.$$

Hence, for all $(\psi, \phi) \in L_p \times L_n$,

$$j^k(\psi \circ W^k_{n,p} \circ \phi) = W^k_{n,p}.$$

By Theorem A.8.9, the manifolds $W^{k,i}_{n,p}$, $1 \leq i \leq i_k$, are disjoint and unique up to order. Hence,

$$\{j^k(\psi \circ W^{k,i}_{n,p} \circ \phi) \ : \ 1 \leq i \leq i_k\}$$

is a permutation of

$$\{W^{k,i}_{n,p} \ : \ 1 \leq i \leq i_k\}.$$

We define an equivalence relation on this set of analytic manifolds. We say

$$W^{k,i}_{n,p} \sim W^{k,i'}_{n,p}$$

if and only if there exists some $(\psi, \phi) \in L_p \times L_n$ such that

$$j^k(\psi \circ W_{n,p}^{k,i} \circ \phi) = W_{n,p}^{k,i'}.$$

Choose one analytic manifold $W_{n,p}^{k,i}$ and consider the subspace of $J_{fibre}^k(E, Y)$ whose projection into each fibre is the union of all the elements in the analytic manifolds $W_{n,p}^{k,i'}$ which are equivalent to $W_{n,p}^{k,i}$, a set we denote by $[W_{n,p}^{k,i}]$.

Using co-ordinate patches for the base B, the fibre X and the manifold Y, we can construct a set of smooth co-ordinate transformations $\{g_{ij}\}$ where

$$g_{ij} \ : \ U_i \cap U_j \ \rightarrow \ L_n^k \times L_p^k$$

and $\{U_i \ : \ i \in \tilde{I}\}$ is an indexed covering of $E \times Y$.

By the Existence Theorem for fibre bundles (see [44, page 14]), there exists a fibre bundle with fibre $[W_{n,p}^{k,i}]$, group $L_n^k \times L_p^k$, base $E \times Y$ and the co-ordinate transformations $\{g_{ij}\}$, $\{U_i \ : \ i \in \tilde{I}\}$. This bundle is unique up to equivalence. By inspection, we see that the total space of this bundle is the restriction of $J_{fibre}^k(E, Y)$ to $[W_{n,p}^{k,i}]$ in each fibre. We shall call the total spaces of these subbundles $N_{n,p}^{k,i}$, $1 \leq i \leq m_k$, where $1 \leq m_k \leq i_k$.

The codimension of each subbundle $N_{n,p}^{k,i}$ is at least the codimension of $W_{n,p}^k$ in $J_{n,p}^{k,0}$ as the codimension of each analytic manifold $W_{n,p}^{k,i}$ is at least the codimension of $W_{n,p}^k$ in $J_{n,p}^{k,0}$. □

Choose $\tilde{k}$ in $\mathbf{N}$ such that the codimension of $W_{n,p}^{\tilde{k}}$ in $J_{n,p}^{\tilde{k},0}$ is greater than $n + m$ which is the dimension of E. This is possible as by Corollary 4.1.6, $W_{n,p}^{\infty}$ is a proalgebraic set of infinite codimension in $J_{n,p}^{\infty,0}$. Then, for every i, $1 \leq i \leq m_{\tilde{k}}$, the codimension of $N_{n,p}^{\tilde{k},i}$ exceeds the dimension of E.

Remark 4.2.2 *Let f be in $C^\infty(E, Y)$. If, for some b in B the germ f_b is of* *infinite type at* e *in* F_b *then*

$$j^k f_b(e) \in \coprod_{i=1}^{m_k} N_{n,p}^{k,i},$$

for any k *in* $\mathbb{N}$.

Remark 4.2.3 *The algebraic set* $W_{n,p}^k = \coprod_{i=1}^{i_k} W_{n,p}^{k,i}$ *is closed in* $J_{n,p}^{k,0}$. *Hence* $\coprod_{i=1}^{m_k} N_{n,p}^{k,i}$ *is closed in* $J_{fibre}^k(E,Y)$.

4.3 The Main Result

Let $\pi : E \to B$ be a smooth fibre bundle with fibre a manifold of dimension n and base B a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Let Y be a smooth p-manifold, $1 \leq p < \infty$, and consider the space of smooth maps $C^\infty(E,Y)$ with the Whitney C^∞ topology. The main result in this chapter is the following theorem.

Theorem 4.3.1 *The subset of* $C^\infty(E,Y)$ *whose elements have germs of codimension less than or equal to* c *everywhere on fibres is open and dense in the Whitney* C^∞ *topology if*

1. $c \geq n + m = \dim E$ *and* $n \geq p \geq 1$ *or if*
2. $c \geq \max\{0, (2n+m-p+1)(p-n+1)-1\}$ *and* $p > n \geq 1$.

The following results are preparation for the proof of Theorem 4.3.1.

Theorem 4.3.2 1. *There exists a dense open subset of* $C^\infty(E,Y)$, *invariant under the actions of the diffeomorphisms of* Y *and the fibre preserving diffeomorphisms of* E, *whose elements are of finite type everywhere on fibres.*

2. *If E is compact, the subset of $C^\infty(E,Y)$ of maps which are of finite type everywhere on fibres is dense and open and invariant under the actions of the diffeomorphisms of Y and the fibre preserving diffeomorphisms of E.*

Proof By Theorem 2.2.13,

$$T_{N^{k,i}_{n,p}} = \{f \in C^\infty(E,Y)\ :\ j^k_{fibre} f \pitchfork N^{k,i}_{n,p}\}$$

is residual in $C^\infty(E,Y)$ in the Whitney C^∞ topology for every k in $\mathbb{N}$ and every i, $1 \le i \le m_k$. For every k in $\mathbb{N}$ let

$$\mathcal{G}^k = \cap_{i=1}^{m_k} T_{N^{k,i}_{n,p}}.$$

Then, for every k in $\mathbb{N}$, $\mathcal{G}^k$ is residual. As $C^\infty(E,Y)$ is a Baire space (see [15, page 44]), $\mathcal{G}^k$ is dense. By Corollary 4.1.6, $W^\infty_{n,p}$ is of infinite codimension in $J^{\infty,0}_{n,p}$. Set

$$\tilde{k} = k(n,m,p) = \min\{k \in \mathbb{N}\ :\ \mathrm{codim} W^k_{n,p} > n+m\}.$$

Then, for every $k \ge \tilde{k}$ and every i, $1 \le i \le m_k$, it is true that

$$T_{N^{k,i}_{n,p}} = \{f \in C^\infty(E,Y)\ :\ j^k_{fibre} f(E) \cap N^{k,i}_{n,p} = \emptyset\}$$

and

$$\mathcal{G}^k = \{f \in C^\infty(E,Y)\ :\ j^k_{fibre} f(E) \cap \coprod_{i=1}^{m_k} N^{k,i}_{n,p} = \emptyset\}.$$

By Proposition 2.2.16 and Remark 4.2.3, $\mathcal{G}^k$ is open for every $k \ge \tilde{k}$. Hence $\mathcal{G}^k$ is open and dense for every $k \ge \tilde{k}$. For any $k \ge \tilde{k}$, $f \in \mathcal{G}^k$ implies f is of finite type everywhere on fibres. Recall that the codimension of a germ is invariant under local co-ordinate changes about its source and target. This proves the first statement of the theorem.

By Lemma 4.3.3 below, if E is compact and f in $C^\infty(E,Y)$ is of finite type everywhere on fibres, then for some k in $\mathbb{N}$, $f \in \mathcal{G}^k$. Let

$$\mathcal{G} = \cup_{k \ge \tilde{k}} \mathcal{G}^k.$$

Then $\mathcal{G}$ is open and dense in $C^\infty(E,Y)$and $f \in \mathcal{G}$ iff f is of finite type everywhere on fibres. □

Lemma 4.3.3 *If E is compact and f in $C^\infty(E,Y)$ is of finite type everywhere on fibres then f is in $\mathcal{G}^k$, for some k in $\mathbb{N}$.*

Proof Assume f is of finite type everywhere on fibres, that E is compact and f is not in $\mathcal{G}^k$, for any k in $\mathbb{N}$. Then there exists an infinite sequence

$$\{e_{k_i} \,:\, i \geq 1,\, e_{k_i} \in E\}$$

such that $f_{\pi(e_{k_i})}$ has a singularity of codimension k_i at e_{k_i} and $k_i > k_j$ whenever $i > j \geq 1$. As E is compact, the sequence contains a subsequence converging to a point, say $\tilde{e}$ in E. The regular points of the fibres F_b, b in B, form an open set in E. Hence $\tilde{e}$ must be a critical point of $f_{\pi(\tilde{e})}$. Hence $f_{\pi(\tilde{e})}$ has a singularity of codimension say $k(\tilde{e}) - 2 \geq 1$ at $\tilde{e}$. Hence

$$j^{k(\tilde{e})}_{fibre} f(\tilde{e}) \notin \coprod_{i=1}^{m_{k(\tilde{e})}} N^{k(\tilde{e}),i}_{n,p} \subseteq J^{k(\tilde{e})}_{fibre}(E,Y).$$

As $\coprod_{i=1}^{m_{k(\tilde{e})}} N^{k(\tilde{e}),i}_{n,p}$ is closed, there exists an open neighbourhood U of $\tilde{e}$ in E such that

$$j^{k(\tilde{e})}_{fibre} f(U) \cap \coprod_{i=1}^{m_{k(\tilde{e})}} N^{k(\tilde{e}),i}_{n,p} = \emptyset.$$

Hence

$$\operatorname{codim}(TKf_{\pi(e)}(e)) \leq k(\tilde{e}) - 2,$$

for every e in U. This contradicts the assumption that f is not in $\mathcal{G}^k$, for any k in $\mathbb{N}$. Hence, if f is of finite type everywhere on fibres and E is compact then f is in $\mathcal{G}^k$, for some k in $\mathbb{N}$. □

Lemma 4.3.4 *A germ f in $E^0_{n,p}$ has a singularity at the origin iff $codim(TKf) \geq |n-p| + 1$.*

Proof 1. Consider the case $n \geq p \geq 1$. The simplest germ which is not a submersion is represented by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}.$$

This germ is of codimension $n - p + 1$. Hence f in $E^0_{n,p}$ has a singularity at the origin iff $\operatorname{codim}(TKf) \geq n - p + 1$.

2. Consider the case $p > n \geq 1$. The simplest germ which is not an immersion is represented by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1^2 \\ x_2 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This germ is of codimension $p - n + 1$. Hence f in $E^0_{n,p}$ has a singularity at the origin iff $\operatorname{codim}(TKf) \geq p - n + 1$. □

Definition 4.3.5 *Let V and W be vector spaces of dimension n and p respectively. Let $q = \min\{n, p\}$. Let $S : V \to W$ be linear, then define $corank(S) = q - rank(S)$. Denote by $Hom(V, W)$ the linear maps from V into W. Define*

$$L^r(V, W) = \{S \in Hom(V, W) \, : \, corank(S) = r\}.$$

Proposition 4.3.6 *The space $L^r(V, W)$ is a submanifold of $Hom(V, W)$ with codimension equal to $(n - q + r)(p - q + r)$.*

For proof see [15, page 60].

Corollary 4.3.7 *1. For all $n \geq p \geq 1$, the codimension of $W^{n-p+2}_{n,p}$ in $J^{n-p+2,0}_{n,p}$ equals $n - p + 1$.*

2. For all $p > n \geq 1$, the codimension of $W_{n,p}^{p-n+2}$ in $J_{n,p}^{p-n+2,0}$ equals $p-n+1$.

Next, for every k in $\mathbf{N}$, upper bounds for the codimension of $W_{n,p}^{n-p+k+1}$, when $n \geq p \geq 1$, and of $W_{n,p}^{k(p-n+1)+1}$, when $p > n \geq 1$, are calculated.

Lemma 4.3.8 *For $n \geq p \geq 1$ and k in $\mathbf{N}$, the codimension of $W_{n,p}^{n-p+k+1}$ in $J_{n,p}^{n-p+k+1,0}$ is at most $n-p+k$.*

Proof We find a submanifold in $J_{n,p}^{n-p+k+1,0}$ of codimension $n-p+k$ contained in $W_{n,p}^{n-p+k+1}$. Let $V_{n,p}^{n-p+k+1}$ be the algebraic subset of $J_{n,p}^{n-p+k+1,0}$ defined by

1. $j^{n-p+2}(V_{n,p}^{n-p+k+1}) \subseteq W_{n,p}^{n-p+2}$ and
2. $a_{(l,0,\ldots,0),1} = 0$, $2 \leq l \leq k$.

The space $W_{n,p}^{n-p+2}$ is the set of all singular $(n-p+2)$-jets in $J_{n,p}^{n-p+2,0}$ and is of codimension $n-p+1$, by the previous corollary. Hence, at any point in $W_{n,p}^{n-p+2}$, the polynomials defining $W_{n,p}^{n-p+2}$ have rank less than or equal to $n-p+1$. The regular locus of $W_{n,p}^{n-p+2}$ is defined locally everywhere by a set of exactly $(n-p+1)$ polynomials which may vary over $W_{n,p}^{n-p+2}$. Choose a in $E_{n,p}^0$. Then

$$da_{\underline{0}} = [a_{ji}]$$

where $a_{ji} = a_{(0,\ldots,0,1,0,\ldots,0),j}$ and the subscript is in the i-th position, $1 \leq i \leq n$ and $1 \leq j \leq p$. The polynomials defining $W_{n,p}^{n-p+2}$ are the determinants of all the $p \times p$ submatrices of $da_{\underline{0}}$.

Consider f in $E_{n,p}^0$ given by

$$f = \begin{pmatrix} x_1^{k+1} + x_2^2 + x_3^2 + \ldots + x_{n-p+1}^2 \\ x_{n-p+2} \\ \vdots \\ x_n \end{pmatrix}.$$

By inspection, codim(TKf) equals $n-p+k$.

Let det(i) be the determinant of the $p \times p$ submatrix of $da_{\underline{0}}$ formed by the i-th column and the last $p-1$ columns, $1 \le i \le n-p+1$. At f in $J^{n-p+k+1,0}_{n,p}$,

$$\frac{\partial \det(i)}{\partial a_{(0,\ldots,0,1,0,\ldots,0),1}} = 1$$

where the subscript is in the i-th position, $1 \le i \le n-p+1$. Hence the polynomials

$$\{\det(i) \,:\, 1 \le i \le n-p+1\}$$

are linearly independent at f. Hence f is in the regular locus of $W^{n-p+2}_{n,p}$ and $W^{n-p+2}_{n,p}$ is defined by the zero set of

$$\{\det(i) \,:\, 1 \le i \le n-p+1\}$$

near f. Hence $V^{n-p+k+1}_{n,p}$ is defined near f by $n-p+k$ linearly independent polynomials. Let U be an open neighbourhood of f in $J^{n-p+k+1,0}_{n,p}$. If U is small enough,

$$\tilde{U} = V^{n-p+k+1}_{n,p} \cap U$$

is a submanifold of $J^{n-p+k+1,0}_{n,p}$ with codimension $n-p+k$. We show that if U is small enough, $\tilde{U} \subseteq W^{n-p+k+1}_{n,p}$.

Choose g in $\tilde{U}$, close to f, then g has the form

$$g(x_1,\ldots,x_n) = \begin{pmatrix} x_2^2 + x_3^2 + \ldots + x_{n-p+1}^2 + g_1^1(x_1,\ldots,x_n) + g_1^2(x_1,\ldots,x_n) \\ x_{n-p+2} + g_{n-p+2}(x_1,\ldots,x_n) \\ \vdots \\ x_n + g_n(x_1,\ldots,x_n) \end{pmatrix}$$

where $\det(i) = 0$, $1 \le i \le n-p+1$, $a_{(l,0,\ldots,0),1} = 0$, $2 \le l \le k$, $g_1^1(x_1,\ldots,x_n)$ is a linear term, $g_1^2(x_1,\ldots,x_n) \in M_n^2$ and $g_i(x_1,\ldots,x_n) \in J^{n-p+k-1,0}_{n,1}$, $n-p+2 \le i \le n$. The projection of $dg_{\underline{0}}$ into

$$\{\underline{y} \in \mathbb{R}^p \,:\, y_1 = 0\}$$

is onto. Hence the linear part of $g_1^1(x_1, \ldots, x_n)$, is a linear combination of the linear parts of $x_i + g_i(x_1, \ldots, x_n)$, $n-p+2 \le i \le n$. Change co-ordinates in $\mathbf{R}^p$ and write g as

$$\begin{pmatrix} x_2^2 + x_3^2 + \ldots + x_{n-p+1}^2 + \tilde{g}_1^2(x_1, \ldots, x_n) \\ x_{n-p+2} + g_{n-p+2}(x_1, \ldots, x_n) \\ \vdots \\ x_n + g_n(x_1, \ldots, x_n) \end{pmatrix}$$

where $\tilde{g}_1^2(x_1, \ldots, x_n) \in M_n^2$. Define

$$\begin{aligned} v_i &= x_i, \quad 1 \le i \le n-p+1, \quad \text{and} \\ v_i &= x_i + g_i(x_1, \ldots, x_n), \quad n-p+2 \le i \le n. \end{aligned}$$

If g is close enough to f then the map

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

is a smooth change of co-ordinates in $\mathbf{R}^n$. In the new co-ordinates,

$$g(v_1, \ldots, v_n) = \begin{pmatrix} v_2^2 + v_3^2 + \ldots + v_{n-p+1}^2 + g^2(v_1, \ldots, v_n) \\ v_{n-p+2} \\ \vdots \\ v_n \end{pmatrix}$$

where $g^2(v_1, \ldots, v_n) \in M_n^2$ and $a_{(l,0,\ldots,0),1} = 0$, $2 \le l \le k$. Consider

$$g_1(v_1, \ldots, v_n) = v_2^2 + v_3^2 + \ldots + v_{n-p+1}^2 + g^2(v_1, \ldots, v_n).$$

Let $\tilde{g}_1(v_2, \ldots, v_{n-p+1})$ be the projection of $g_1(v_1, \ldots, v_n)$ onto $J_{n-p,1}^{n-p+k+1,0}$. We may write

$$g_1(v_1, \ldots, v_n) = \tilde{g}_1(v_2, \ldots, v_{n-p+1}) + \tilde{\tilde{g}}(v_1, \ldots, v_n)$$

where $\tilde{\tilde{g}}(v_1, \ldots, v_n) \in M_n^2$ is a linear combination of monomials $v_1^{I_1} \ldots v_n^{I_n}$ such that

$$\Sigma_{i=n-p+2}^n I_i + I_1 \ge 1.$$

Now $\tilde{g}_1(v_2, \ldots, v_{n-p+1})$ is Morse if g is close enough to f. Hence we can find a smooth change of co-ordinates in these $n-p$ variables so that

$$\tilde{g}_1(v_2, \ldots, v_{n-p+1}) = v_2^2 + v_3^2 + \ldots + v_{n-p+1}^2.$$

Hence, up to smooth changes of co-ordinates in $\mathbf{R}^n$ and $\mathbf{R}^p$ we may write $g(x_1, \ldots, x_n)$ as

$$\begin{pmatrix} x_2^2 + x_3^2 + \ldots + x_{n-p+1}^2 + \tilde{g}(x_1, \ldots, x_n) \\ x_{n-p+2} \\ \vdots \\ x_n \end{pmatrix}$$

where $a_{(l,0,\ldots,0),1} = 0$, $2 \leq l \leq k$, and where $\tilde{g}(x_1, \ldots, x_n)$ in M_n^2 is a linear combination of monomials $x_1^{I_1} \ldots x_n^{I_n}$ such that

$$\Sigma_{i=n-p+2}^n I_i + I_1 \geq 1.$$

By inspection,

$$\dim J_{n,p}^{n-p+k+1,0} / TK^{n-p+k+1} g \geq n-p+k.$$

Hence $g \in W_{n,p}^{n-p+k+1}$. Hence, if U is small enough, $\tilde{U} \subseteq W_{n,p}^{n-p+k+1}$. Hence $\operatorname{codim}(W_{n,p}^{n-p+k+1}) \leq n-p+k$. □

Lemma 4.3.9 *For $p > n \geq 1$ and k in $\mathbf{N}$, the codimension of $W_{n,p}^{k(p-n+1)+1}$ in $J_{n,p}^{k(p-n+1)+1,0}$ is at most $pk-n+1$.*

Proof We find a submanifold in $J_{n,p}^{k(p-n+1)+1,0}$ of codimension $pk-n+1$ contained in $W_{n,p}^{k(p-n+1)+1}$. Let $V_{n,p}^{k(p-n+1)+1}$ be the algebraic subset of $J_{n,p}^{k(p-n+1)+1,0}$ defined by

1. $j^{p-n+2}(V_{n,p}^{k(p-n+1)+1}) \subseteq W_{n,p}^{p-n+2}$ and

2. $a_{(l,0,\ldots,0),j} = 0$, $1 \leq j \leq p$, $2 \leq l \leq k$.

The space $W^{p-n+2}_{n,p}$ is the set of all singular $(p-n+2)$-jets in $J^{p-n+2,0}_{n,p}$ and is of codimension $p-n+1$, by Corollary 4.3.7. Hence, at any point in $W^{p-n+2}_{n,p}$, the polynomials defining $W^{p-n+2}_{n,p}$ have rank less than or equal to $p-n+1$. The regular locus of $W^{p-n+2}_{n,p}$ is defined locally everywhere by a set of exactly $(p-n+1)$ polynomials which may vary over $W^{p-n+2}_{n,p}$. Choose a in $E^0_{n,p}$. Then

$$da_{\underline{0}} = [a_{ji}]$$

where $a_{ji} = a_{(0,\ldots,0,1,0,\ldots,0),j}$ and the subscript is in the i-th position, $1 \le i \le n$, $1 \le j \le p$. The polynomials defining $W^{p-n+2}_{n,p}$ are the determinants of all the $n \times n$ submatrices of $da_{\underline{0}}$.

Consider f in $E^0_{n,p}$ given by

$$f = \begin{pmatrix} x_1^{k+1} \\ x_2 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By inspection, codim(TKf) equals $k(p-n+1)$.

Let $\det(j)$ be the determinant of the $n \times n$ submatrix of $da_{\underline{0}}$ formed by rows 2,3, $\ldots$, n and row j, where $n+1 \le j \le p$ or $j=1$. At f in $J^{k(p-n+1)+1,0}_{n,p}$,

$$\frac{\partial \det(j)}{\partial a_{(1,0,\ldots,0),j}} = 1,$$

$n+1 \le j \le p$ or $j=1$. Hence the polynomials

$$\{\det(j) \,:\, n+1 \le j \le p \text{ or } j=1\}$$

are linearly independent at f. Hence $j^{p-n+2}f$ is in the regular locus of $W^{p-n+2}_{n,p}$ and $W^{p-n+2}_{n,p}$ is defined by the zero set of

$$\{\det(j) \,:\, n+1 \le j \le p \text{ or } j=1\}$$

near $j^{p-n+2}f$. Hence $V_{n,p}^{k(p-n+1)+1}$ is defined near f by $(p-n+1)+(k-1)p = pk-n+1$ linearly independent polynomials. Let U be an open neighbourhood of f in $J_{n,p}^{k(p-n+1)+1,0}$. If U is small enough,

$$\tilde{U} = V_{n,p}^{k(p-n+1)+1} \cap U$$

is a submanifold of $J_{n,p}^{k(p-n+1)+1,0}$ with codimension $pk-n+1$. We show that if U is small enough, $\tilde{U} \subseteq W_{n,p}^{k(p-n+1)+1}$.

Choose g in $\tilde{U}$, close to f, then g has the form

$$g(x_1,\ldots,x_n) = \begin{pmatrix} g_1(x_1,\ldots,x_n) \\ \vdots \\ g_p(x_1,\ldots,x_n) \end{pmatrix}$$

where $\det(j) = 0$, $n+1 \leq j \leq p$ or $j = 1$, and $a_{(l,0,\ldots,0),j} = 0$, $1 \leq j \leq p$, $2 \leq l \leq k$. The projection of $dg_{\underline{0}}$ into

$$\{\underline{y} \in \mathbf{R}^p \,:\, y_j = 0,\, j = 1, n+1, \ldots, p\}$$

is onto. Hence the linear part of g_j, $j = 1, n+1, \ldots, p$ is a linear combination of the linear parts of $g_2, \ldots, g_n$. We change co-ordinates in $\mathbf{R}^p$ and write g as

$$\begin{pmatrix} \tilde{g}_1(x_1,\ldots,x_n) \\ \vdots \\ \tilde{g}_p(x_1,\ldots,x_n) \end{pmatrix}$$

where $\tilde{g}_j \in M_n^2$, $j = 1, n+1, \ldots, p$. Define

$$\begin{aligned} v_i &= \tilde{g}_i(x_1,\ldots,x_n), \quad 2 \leq i \leq n, \quad \text{and} \\ v_1 &= x_1. \end{aligned}$$

If g is close enough to f then the map

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

is a smooth change of co-ordinates in $\mathbf{R}^n$. In the new co-ordinates,

$$g(v_1,\ldots,v_n) = \begin{pmatrix} \tilde{g'}_1(v_1,\ldots,v_n) \\ v_2 \\ \vdots \\ v_n \\ \tilde{g'}_{n+1}(v_1,\ldots,v_n) \\ \vdots \\ \tilde{g'}_p(v_1,\ldots,v_n) \end{pmatrix}$$

where $\tilde{g'}_j(v_1,\ldots,v_n) \in M_n^2$, $j = 1, n+1, \ldots, p$ and $a_{(l,0,\ldots,0),j} = 0$, $1 \le j \le p$, $2 \le l \le k$. By inspection,

$$\dim J_{n,p}^{k(p-n+1)+1,0}/TK^{k(p-n+1)+1}g \ge k(p-n+1).$$

Hence $g \in W_{n,p}^{k(p-n+1)+1}$. Hence, if U is small enough, $\tilde{U} \subseteq W_{n,p}^{k(p-n+1)+1}$. Hence $\operatorname{codim}(W_{n,p}^{k(p-n+1)+1}) \le pk - n + 1$. □

Next, for every k in $\mathbf{N}$, lower bounds for the codimension of $W_{n,p}^{n-p+k+1}$, when $n \ge p \ge 1$, and for the codimension of $W_{n,p}^{k(p-n+1)+1}$, when $p > n \ge 1$, are calculated.

Lemma 4.3.10 *Let*

$$A = [a_{ij}]$$

be a $q \times q$ matrix, q in $\mathbf{N}$. Assume the a_{ij} are differentiable (C^1) functions of the real variable t. Let A_{ij} be the determinant of the matrix formed from A by deleting the i-th row and j-th column, $1 \le i,j \le q$. If every $A_{ij} = 0$, $1 \le i,j \le q$, at $t = \tilde{t}$, then $\frac{d(\det(A))}{dt} = 0$ at $t = \tilde{t}$.

For proof see [2, page 140].

Lemma 4.3.11 *1. If $n \ge p \ge 1$ and k is in $\mathbf{N}$ then there exists a germ in $E_{n,p}^0$ with a singularity of codimension $n-p+k$ at the origin.*

2. If $p > n \geq 1$ and k is in $\mathbf{N}$ then there exists a germ in $E^0_{n,p}$ with a singularity of codimension $k(p-n+1)$ at the origin.

Proof 1. Consider

$$f = \begin{pmatrix} x_1^{k+1} + x_2^2 + x_3^2 + \ldots + x_{n-p+1}^2 \\ x_{n-p+2} \\ \vdots \\ x_n \end{pmatrix}.$$

By inspection, $\operatorname{codim}(TKf) = n - p + k$.

2. Consider

$$f = \begin{pmatrix} x_1^{k+1} \\ x_2 \\ \vdots \\ x_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By inspection, $\operatorname{codim}(TKf) = k(p-n+1)$. □

Lemma 4.3.12 *1. Suppose g is a regular point in an irreducible component of $W^{n-p+k+1}_{n,p} \subseteq J^{n-p+k+1,0}_{n,p}$ of highest dimension, where $n \geq p \geq 1$ and k is in $\mathbf{N}$. Then codim$(TKg) = n - p + k$.*

2. Suppose g is a regular point in an irreducible component of $W^{k(p-n+1)+1}_{n,p} \subseteq J^{k(p-n+1)+1,0}_{n,p}$ of highest dimension, where $p > n \geq 1$ and k is in $\mathbf{N}$. Then codim$(TKg) = k(p-n+1)$.

Proof 1. The algebraic set $W^{n-p+k+1}_{n,p}$ is defined by a set of polynomials in the indeterminates

$$\{a_{I,j}\}$$

where $I = (I_1, \ldots, I_n)$, $I_i \in \mathbf{Z}$, $1 \leq i \leq n$, $0 \leq I_i \leq n - p + k + 1$, $1 \leq \Sigma_{i=1}^n I_i \leq n - p + k + 1$ and $1 \leq j \leq p$.

These polynomials are the determinants of all square submatrices of codimension $n-p+k-1$ in the matrix representation of the linear map $\beta(g)$ defined in the proof of Lemma 4.1.3. Each one of the regular loci of the irreducible components of highest dimension in $W_{n,p}^{n-p+k+1}$ is defined locally by a set of linearly independent polynomials. Hence one of the determinants defining the regular locus near g must have a non-zero derivative with respect to one of the indeterminates $a_{I,j}$.

By Lemma 4.3.10, one of the square submatrices of codimension $n-p+k$ in the matrix representation of $\beta(g)$ has a non-zero determinant. Hence

$$\dim J_{n,p}^{n-p+k+1,0}/TK^{n-p+k+1}g = n-p+k.$$

By Nakayama's Lemma (A.7), setting $\mathcal{R} = E_n$, $\mathcal{K} = \mathbf{R}$, $\mathcal{M} = M_n$, $C = E_{n,p}^0$, $A = TKg$ and $d = n-p+k$,

$$M_{n,p}^{n-p+k+1} \subseteq TKg.$$

Hence

$$\dim E_{n,p}^0/TKg = n-p+k.$$

2. If $p > n \geq 1$ then there may not be germs of every codimension. By Lemma 4.3.11 there exist singular germs in $E_{n,p}^0$ of codimension $k(p-n+1)$ for every $p > n \geq 1$ and for every k in $\mathbf{N}$. The determinants of the square submatrices of codimension $k(p-n+1)-1$ in the matrix representation of $\beta(g)$ with respect to $W_{n,p}^{k(p-n+1)+1}$ define the jets of codimension $\geq k(p-n+1)$ in $J_{n,p}^{k(p-n+1)+1,0}$. The proof is completed as above.

If there exists no jet of codimension $l-1$ in $W_{n,p}^l$, $l > p-n+2$, the set $W_{n,p}^l$ is defined by the zero set of square submatrices of codimension $\geq l-1$. In this case the application of Nakayama's Lemma fails to give a useful result. □

Lemma 4.3.13 *1. The codimension of $W_{n,p}^{n-p+k+1}$ in $J_{n,p}^{n-p+k+1,0}$ is at least $n-p+k$ if $n \geq p \geq 1$ and k is in $\mathbf{N}$.*

2. The codimension of $W_{n,p}^{k(p-n+1)+1}$ in $J_{n,p}^{k(p-n+1)+1,0}$ is at least $p-n+k$ if $p > n \geq 1$ and k is in $\mathbf{N}$.

Proof We prove the first case, the second is similar. Let $l = n-p+k+1$. Let

$$V_1^l, \dots, V_{i_l}^l$$

be the irreducible components of $W_{n,p}^l$. By Theorem A.8.11,

$$(\pi_l^{l+1})^{-1}(V_i^l)$$

is an irreducible algebraic set in $J_{n,p}^{l+1,0}$, $1 \leq i \leq i_l$. Let

$$V_1^{l+1}, \dots, V_{i_{l+1}}^{l+1}$$

be the irreducible components of $W_{n,p}^{l+1}$. By Lemma 4.1.3

$$W_{n,p}^{l+1} \subseteq (\pi_l^{l+1})^{-1} W_{n,p}^l.$$

Hence

$$V_i^{l+1} \subseteq \cup_{j=1}^{i_l}((\pi_l^{l+1})^{-1}(V_j^l))$$

and so

$$V_i^{l+1} \subseteq \cup_{j=1}^{i_l}(V_i^{l+1} \cap ((\pi_l^{l+1})^{-1}(V_j^l)))$$

each i, $1 \leq i \leq i_{l+1}$. An irreducible algebraic set cannot be decomposed into smaller algebraic sets. Hence for each i, $1 \leq i \leq i_{l+1}$,

$$V_i^{l+1} \subseteq (\pi_l^{l+1})^{-1}(V_j^l)$$

for some j, $1 \leq j \leq i_l$. By Lemma A.8.6 and Theorem A.8.11 either

1. $V_i^{l+1} = (\pi_l^{l+1})^{-1}(V_j^l)$ or

2. $\operatorname{codim}(V_i^{l+1}) > \operatorname{codim}(\pi_l^{l+1})^{-1}(V_j^l) = \operatorname{codim}(V_j^l)$.

Let V_j^l be an irreducible component of $W_{n,p}^l$ of highest dimension. By Definition A.8.7, the regular locus of V_j^l is non-empty. By Lemma 4.3.12,

$$(\pi_l^{l+1})^{-1}(V_j^l) \not\subset W_{n,p}^{l+1}.$$

Hence

$$\operatorname{codim}(V_i^{l+1}) > \operatorname{codim} W_{n,p}^l,$$

$1 \leq i \leq i_{l+1}$. Hence

$$\operatorname{codim} W_{n,p}^{l+1} > \operatorname{codim} W_{n,p}^l.$$

By Corollary 4.3.7,

$$\operatorname{codim} W_{n,p}^{n-p+2} = n - p + 1.$$

By induction,

$$\operatorname{codim} W_{n,p}^{n-p+k+1} \geq n - p + k.$$

□

Lemma 4.3.14 *1. If $n \geq p \geq 1$ and k is in $\mathbf{N}$ then*

$$codim W_{n,p}^{n-p+k+1} = n - p + k.$$

2. If $p > n \geq 1$ and k is in $\mathbf{N}$ then

$$p - n + k \leq codim W_{n,p}^{k(p-n+1)+1} \leq pk - n + 1 = p - n + k + (p-1)(k-1).$$

Lemma 4.3.14 is a direct consequence of Lemmas 4.3.8, 4.3.9 and 4.3.13.

Proof of Theorem 4.3.1. 1. By the proof of Theorem 4.3.2 and Lemma 4.3.14, $\mathcal{G}^{c+2}$ is dense and open in $C^\infty(E, Y)$ in the Whitney C^∞ topology if $c \geq n + m$. Recall, $f \in \mathcal{G}^{c+2}$, for $c \geq n + m$, iff the germ of f is of codimension not exceeding c everywhere on fibres.

2. By the proof of Theorem 4.3.2 and Lemma 4.3.14, $\mathcal{G}^{c+2}$ is dense and open in $C^\infty(E, Y)$ in the Whitney C^∞ topology if $c \geq k(p-n+1)-1$ and $n+m < p-n+k$. Recall, for $c \geq k(p-n+1)-1$ and $n+m < p-n+k$, that is, for $c \geq \max\{0, (2n+m-p+1)(p-n+1)-1\}$, $f \in \mathcal{G}^{c+2}$ iff the germ of f is of codimension not exceeding c everywhere on fibres. □

Corollary 4.3.15 *There is an open dense subset of $C^\infty(E, Y)$ whose elements are d-K determined everywhere on fibres if*

1. $d \geq m+n+1$ *and* $n \geq p \geq 1$ *or if*
2. $d \geq \max\{1, (2n+m-p+1)(p-n+1)\}$ *and* $p > n \geq 1$.

Proof If $\operatorname{codim}(TKf) \leq k$ then

$$M_{n,p}^{k+1} \subseteq TKf$$

and so, by Theorem 3.3.3, f is $(k+1)$-K determined. The proof is finished by applying Theorem 4.3.1. □

Corollary 4.3.16 *If $p \geq 2n+m$, the subset of $C^\infty(E, Y)$ whose elements are immersions on fibres is open and dense.*

This is the Whitney Immersion Theorem for bundles. It is a direct consequence of Theorem 4.3.1 and Lemma 4.3.4.

4.4 Isolated Critical Points

Let $\pi : E \to B$ be a smooth fibre bundle with fibre a manifold of dimension n and base B a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Let Y be a smooth p-manifold, $1 \leq p < \infty$, and consider the space of smooth maps $C^\infty(E, Y)$ with the Whitney C^∞ topology.

Definition 4.4.1 *For f in $C^\infty(E,Y)$ and y in Y, the preimage $f^{-1}(y)$ is a level set of f.*

Theorem 4.4.2 *1. There exists a dense open subset of $C^\infty(E,Y)$, invariant under the actions of the diffeomorphisms of Y and the fibre preserving diffeomorphisms of E, such that on fibres, for any element, each critical point is isolated in the level set of the corresponding critical value.*

2. If $\dim(Y) = p = 1$, there exists a dense open subset of $C^\infty(E,Y)$, invariant under the actions of the diffeomorphisms of Y and the fibre preserving diffeomorphisms of E, such that on fibres, for any element, the critical points are all isolated.

Theorem 4.4.2 is an immediate consequence of the following local result, namely Theorem 4.4.3, and the global result Theorem 4.3.1.

Theorem 4.4.3 *1. If f in $E^0_{n,p}$ is of finite type, then either $\underline{0} \notin \Sigma f$, the set of critical points of f, or $\underline{0}$ is isolated as a critical point in the germ $f^{-1}(\underline{0})$.*

2. If f in E^0_n is of finite type, then either $\underline{0} \notin \Sigma f$ or $\underline{0}$ is an isolated critical point of f.

Theorem 4.4.3 follows from Theorem 4.4.4, Theorem 4.4.5 and the technical Lemma 4.4.6. We prove Theorems 4.4.3 and 4.4.4 later.

Theorem 4.4.4 *A holomorphic germ $f : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}^p, \underline{0})$ is of finite type iff $\underline{0} \notin \Sigma f$ or $\underline{0}$ is isolated as a critical point in $f^{-1}(\underline{0})$.*

Theorem 4.4.5 *If f is a real analytic germ, then the following are equivalent:*

1. f is of finite type as a real analytic germ

2. f is of finite type as a smooth germ

3. the complexification of f is of finite type as a holomorphic germ.

For proof, see [48, page 491].

Lemma 4.4.6 *Suppose U is an open subset of $\mathbf{R}^n$ containing the origin and $f : (U, \underline{0}) \to (\mathbf{R}^p, \underline{0})$ is a smooth germ. Suppose a germ $g : (U, \underline{0}) \to (\mathbf{R}^p, \underline{0})$ is contact equivalent to f. Then the germ $g^{-1}(\underline{0}) \cap \Sigma g$ is diffeomorphic to the germ $f^{-1}(\underline{0}) \cap \Sigma f$.*

Proof As f and g are contact equivalent there exists some ϕ in $K_{n,p}$, the contact group, such that $g = \phi.f$. Using the notation introduced in §3.2 we write

$$\phi(\underline{x}, \underline{y}) = (h(\underline{x}), \psi(\underline{x}, \underline{y}))$$

where $\underline{x} \in U$, $\underline{y} \in \mathbf{R}^p$, $h \in L_n$, $\psi_{\underline{0}} \in L_p$ and $\psi(\underline{x}, \underline{0}) = \underline{0}$ for all $\underline{x}$.

By definition,

$$\phi.f \circ h(\underline{x}) = \psi(\underline{x}, f(\underline{x})).$$

Hence

$$d(\phi.f \circ h)_{\underline{x}} = [\tfrac{\partial \psi_s}{\partial x_i}]_{(\underline{x}, f(\underline{x}))} + [\tfrac{\partial \psi_s}{\partial y_j}]_{(\underline{x}, f(\underline{x}))} \circ df_{\underline{x}}.$$

Since $\psi(\underline{x}, \underline{0}) = \underline{0}$ for all $\underline{x}$. we have

$$\tfrac{\partial \psi_s}{\partial x_i} = 0$$

for every i, $1 \leq i \leq n$, and for every s, $1 \leq s \leq p$, at $(\underline{x}, \underline{0})$. Since $\psi_{\underline{0}} \in L_p$ the matrix

$$[\tfrac{\partial \psi_i}{\partial y_j}]_{(\underline{x}, f(\underline{x}))}$$

has full rank for $\underline{x}$ near $\underline{0}$. Hence if $\underline{x}' \in U$ and $f(\underline{x}') = \underline{0}$ then

$$d(\phi.f \circ h)_{\underline{x}'} = [\tfrac{\partial \psi_s}{\partial y_j}]_{(\underline{x}',\underline{0})} \circ df_{\underline{x}'}.$$

Hence $h(\underline{x}')$ is a critical point of $\phi.f$ iff $\underline{x}'$ is a critical point of f. Hence the germ $h(f^{-1}(\underline{0}) \cap \Sigma f)$ is equal to the germ $(\phi.f)^{-1}(\underline{0}) \cap \Sigma\phi.f$. □

Theorem 4.4.4 is proved implicitly in Wall's article [48, page 492] but not stated explicitly. We outline his method of proof below.

Definition 4.4.7 *For f in $E^0_{n,p}$ the subspace T_eKf of $E_{n,p}$ is defined by*

$$T_eKf = J(f) + f^*M_pE_{n,p}.$$

The subspace T_eKf is an E_n-submodule of $E_{n,p}$, closely related to $TK(f)$. It is called the extended tangent space to the contact orbit of f (see [48]).

Definition 4.4.8 *The germ f in $E^0_{n,p}$ is K stable iff $T_eKf = E_{n,p}$.*

Lemma 4.4.9 *Let f in $E^0_{n,p}$ be a smooth germ defined on an open set U and let $\underline{x}$ be in $U \backslash \{\underline{0}\}$. Then the germ of f at $\underline{x}$ is K unstable iff $df_{\underline{x}}$ is not surjective and $f(\underline{x}) = \underline{0}$.*

Proof If $f(\underline{x}) \neq \underline{0}$, then $f_j(\underline{x}) \neq 0$, some j, $1 \leq j \leq p$.
Hence $f_j(\underline{x})$ is invertible in E_n.
Hence $T_eKf = E_{n,p}$.
If $f(\underline{x}) = \underline{0}$, then $f^*M_p \subseteq M_n$ so

$$E_{n,p}\,/f^*M_p\,E_{n,p} \text{ projects onto } E_{n,p}\,/M_n\,E_{n,p} \cong \mathbf{R}^p.$$

If $df_{\underline{x}}$ is surjective, $J(f) = E_{n,p}$.

If $df_{\underline{x}}$ is not surjective, $J(f)$ does not include all germs with non-zero constants. Hence

$$J(f) + f^* M_p E_{n,p} = T_e K f \neq E_{n,p}.$$

□

The lemma above (see [48, page 492]) generalizes immediately to the real analytic and holomorphic cases. For the moment, consider holomorphic germs.

Theorem 4.4.10 *A holomorphic germ $f : (\mathbf{C}^n, \underline{0}) \to (\mathbf{C}^p, \underline{0})$ is of finite type iff there is a neighbourhood U of $\underline{0}$ in $\mathbf{C}^n$ such that for every $\underline{z}$ in $U \backslash \{\underline{0}\}$, the germ of f at $\underline{z}$ is K stable.*

For proof see [48, page 492].

Theorem 4.4.4 follows immediately from Lemma 4.4.9 and Theorem 4.4.10. It provides a useful characterization of holomorphic germs. It remains to prove Theorem 4.4.3.

Proof of Theorem 4.4.3. 1. Assume that f in $E^0_{n,p}$ is of finite type and $\underline{0} \in \Sigma f$. By Corollary 3.3.4, f is contact equivalent to $j^k f$, for some k in $\mathbf{N}$. By Lemma 4.4.6, the germ $(j^k f)^{-1}(\underline{0}) \cap \Sigma j^k f$ is diffeomorphic to the germ $f^{-1}(\underline{0}) \cap \Sigma f$. The germ $j^k f$ is real analytic and of finite type. By Theorem 4.4.5, its complexification $j^k \tilde{f}$ is a holomorphic germ of finite type. By Theorem 4.4.4, the germ

$$(j^k \tilde{f})^{-1}(\underline{0}) \cap \Sigma j^k \tilde{f} = \{\underline{0}\}.$$

Hence, the germ

$$(j^k f)^{-1}(\underline{0}) \cap \Sigma j^k f = \{\underline{0}\}.$$

Hence, the germ

$$f^{-1}(\underline{0}) \cap \Sigma f = \{\underline{0}\}.$$

2. Assume that f in E_n^0 is of finite type, that $\underline{0}$ is a critical point of f and that k in $\mathbb{N}$ is large enough so that f is contact equivalent to $j^k f$. As above, the complexification $j^k\tilde{f} : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ is a holomorphic germ of finite type. Now, $\Sigma j^k \tilde{f}$ is an analytic germ of dimension one or zero. If the dimension is one then $j^k\tilde{f}$ is locally zero. In this case, the germ $j^k f$ is locally zero and hence is not of finite type. Hence $\underline{0}$ is an isolated critical point of $j^k\tilde{f}$. Hence $\underline{0}$ is an isolated critical point of $j^k f$ and of f. □

The following lemma relates the codimension of T_eKf in $E_{n,p}$ to the codimension of TKf in $E^0_{n,p}$ and is included for the sake of completeness.

Lemma 4.4.11 *Suppose $f \in E^0_{n,p}$, $n \geq 1$ and $p \geq 1$.*

1. *If the germ f is a submersion at the origin then*

$$codim(TKf) = codim(T_eKf) = 0.$$

2. *If the germ f is not a submersion at the origin and f is of finite type then*

$$codim(TKf) = codim(T_eKf) + (n - p).$$

3. *If the germ f is of infinite type at the origin then*

$$codim(TKf) = codim(T_eKf) = \infty.$$

The first statement follows immediately from the definitions. The rest is proved in Wall's article [48, page 509].

By the lemma above and the remarks in §3.2, for any f in $E^0_{n,p}$, the codimension of T_eKf in $E_{n,p}$ is invariant under local changes of co-ordinates about the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$. Hence $\text{codim}(T_eKf)$ may be used in place of $\text{codim}(TKf)$ to measure the complexity of germs of smooth maps between manifolds. An immediate consequence of Theorem 4.3.1 and Lemma 4.4.11 is the theorem below.

Theorem 4.4.12 *The set of all elements f in $C^\infty(E, Y)$ such that the $\text{codim}(T_eKf_b(e)) \leq c$ everywhere is open and dense if*

1. $c \geq m + p$ *and* $n \geq p \geq 1$ *or*
2. $c \geq \max\{p - n, (2n + m - p + 2)(p - n + 1) - 2\}$ *and* $p > n \geq 1$.

In a sense, the use of $\text{codim}(TKf)$ is more natural than the use of $\text{codim}(T_eKf)$ since f in $E^0_{n,p}$ is the germ of a submersion or an immersion iff $\text{codim}(TKf)$ is zero. By the second statement of Lemma 4.4.11, this does not hold for $\text{codim}(T_eKf)$.

4.5 The A_k Singularities

Definition 4.5.1 *Let X be a smooth n-manifold where $n \geq 1$, let Y be a smooth 1-manifold and let $f : X \to Y$ be a smooth map.*

Then f has an A_k singularity at x in X, if in some local co-ordinates mapping x to the origin in $\mathbb{R}^n$, f may be written

$$f(x_1, \ldots, x_n) = f(x) \pm x_1^{k+1} - x_2^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2,$$

where $1 \leq i \leq n$ and $k \in \mathbb{N}$.

We call A_1 singularities Morse singularities and A_2 singularities "birth-death" singularities after Igusa [23]. The name "birth-death" singularity derives from the following observation.

Define

$$\begin{array}{rcll} f_t & : & \mathbf{R} \to \mathbf{R} & \text{by} \\ & & x \mapsto x^3 - 3tx, & t \in \mathbf{R}. \end{array}$$

Then f_t is a smooth cubic function with critical points $\pm\sqrt{t}$ when $t \geq 0$. If t is strictly positive, then f_t has a nondegenerate maximum at $x = -\sqrt{t}$ and a nondegenerate minimum at $x = +\sqrt{t}$. The map f_0 has a point of inflection at $x = 0$ and no other critical points. If t is strictly negative, then f_t has no critical points. Consider the graph of f_t, t in $\mathbf{R}$. As t approaches zero from above, the maximum and minimum approach the origin in $\mathbf{R}^2$. The "death" of a pair of Morse singularities with indices which differ by one results, in this case, in the appearance of an A_2 singularity. Similarly, the perturbation of an A_2 singularity may result in the "birth" of a pair of Morse singularities with indices which differ by one. Morse functions on the circle may be perturbed into others with more or fewer pairs of nondegenerate maxima and minima by the introduction of singularities of type A_2 only (see Figures 6.2, 6.3 and 6.4).

Example 4.5.2 *Let f in $E^0_{n,1}$ be defined by*

$$f(\underline{x}) = \pm x_1^{k+1} \pm x_2^2 \pm \ldots \pm x_n^2$$

for some k in $\mathbf{N}$. Then $J(f) = < x_1^k, x_2, \ldots, x_n >$ and $TKf = M_n J(f)$. Hence codim$(TKf) = k + n - 1$ and codim$(T_e Kf) = k$.

Corollary 4.5.3 *Let $\pi : E \to B$ be a smooth fibre bundle with fibre a 1-manifold and base B a manifold of dimension m, $0 \leq m < \infty$. Let Y be a smooth 1-manifold. Then the set of all smooth maps from E into Y whose singularities on fibres are only of types $A_1, \ldots, A_{m+1}$ is open and dense in the Whitney C^∞ topology.*

Corollary 4.5.3 is an immediate consequence of Theorem 4.3.1.

4.6 Parametrized Generalized Morse Functions

Definition 4.6.1 *Let X be a smooth n-manifold, where $n \geq 1$, let Y be a smooth 1-manifold and let $f : X \to Y$ be a smooth map. Following Igusa [23], we call the map f a generalized Morse function if every critical point of f is either of type A_1 or of type A_2.*
In addition, if $\pi : E \to B$ is a smooth fibre bundle and $g : E \to Y$ is a smooth map then we call the map g a parametrized generalized Morse function if g is a generalized Morse function on fibres.

Lemma 4.6.2 *For all n in $\mathbb{N}$ and singular germs g in E_n^0,*

1. *$codim(TKg) = n$ iff g is Morse at the origin and*
2. *the subset $W_{n,1}^{n+2}$ is equal to*

$$\{g \in J_{n,1}^{n+2,0} \ : \ g \text{ has a degenerate critical point at the origin}\}.$$

Proof 1. For g in E_n^0, $\text{codim}(TKg) = n$ iff $TKg \oplus \text{Span}\{x_1, \ldots, x_n\} = E_n^0$ iff $J(f) =< x_1, \ldots, x_n >$ iff g is Morse at the origin.

2. By the proof of step 1 above, g in E_n^0 has a degenerate critical point at the origin iff $\text{codim}(TKg) \geq n + 1$. □

Lemma 4.6.3 *The germ g in E_n^0 has an A_2 singularity at the origin iff $j^{n+2}g$ is in the regular loci of the irreducible components of the highest dimension of the algebraic set $W_{n,1}^{n+2} \subset J_{n,1}^{n+2,0}$.*

Proof 1. Assume that g has an A_2 singularity at the origin. Then, in some choice of co-ordinates,

$$g(x_1, \ldots, x_n) = x_1^3 \pm x_2^2 \pm \ldots \pm x_n^2.$$

By Lemma 4.6.2, the set $W^{n+2}_{n,1}$ is equal to

$$\{a \in J^{n+2,0}_{n,1} \ : \ a \text{ has a degenerate critical point at the origin}\}.$$

Hence $W^{n+2}_{n,1}$ is defined globally by the set of $(n+1)$ equations

$$a_{(1,0,\ldots,0)} = a_{(0,1,0,\ldots,0)} = \ldots = a_{(0,\ldots,0,1)} = 0$$

where the subscript is in the i-th position, $1 \leq i \leq n$, and

$$\det(Ha)_{\underline{0}} = 0$$

where $(Ha)_{\underline{0}}$ is the Hessian of a in $J^{n+2,0}_{n,1}$ evaluated at the origin. Now,

$$\frac{\partial \det(Ha)_{\underline{0}}}{\partial a_{(2,0,\ldots,0)}} \neq 0$$

when a equals g. Hence the polynomials defining $W^{n+2}_{n,1}$ have rank $(n+1)$ at g. Hence g is in the regular loci of the irreducible components of the highest dimension of $W^{n+2}_{n,1}$.

2. Assume that $g \in E^0_n$ and $j^{n+2}g$ is in the regular loci of the irreducible components of the highest dimension of $W^{n+2}_{n,1}$. By Lemma 4.3.12,

$$\operatorname{codim}(TKg) = n+1.$$

Suppose $f \in E^0_n$ then, by Lemma 4.6.2, $\operatorname{codim}(TKf) = n$ iff

$$E^0_n = TKf \oplus \operatorname{Span}\{x_1, \ldots, x_n\}.$$

Hence, $\operatorname{codim}(TKf) = n+1$ iff

$$E^0_n = TKf \oplus \operatorname{Span}\{x_1, \ldots, x_n, a\}$$

where a is a homogeneous polynomial of degree two. Hence

$$M^3_n \subseteq TKg.$$

By Theorem 3.3.3, g is 3-K determined. Hence, by changing co-ordinates, we can eliminate all terms of degree greater than three in a Taylor expansion for g.

As $j^{n+2}g$ is in the regular loci of the irreducible components of the highest dimension of $W^{n+2}_{n,1}$,

$$\frac{\partial \det(Hg)_0}{\partial b} \neq 0$$

for some coefficient b so by Lemma 4.3.10, the Hessian $(Hg)_{\underline{0}}$ has rank $n-1$. Hence, in some local co-ordinates we may write

$$g(x_1, \ldots, x_n) = \pm x_2^2 \pm \ldots \pm x_n^2 + x_1 h(x_1, \ldots, x_n)$$

where $h(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree two. We may eliminate all terms in h which are not multiples of x_1^2 by changing co-ordinates and possibly introducing strictly higher order multiples of x_1^2 to the expression for g. For example, using the transformation

$$x_2 \mapsto x_2 + \frac{1}{2}x_1^2$$

we may replace $x_2^2 + x_2 x_1^2$ by $x_2^2 - \frac{1}{4}x_1^4$ and by using the transformation

$$x_2 \mapsto x_2 + \frac{1}{2}x_1 x_3$$

we may replace $x_2^2 + x_1 x_2 x_3$ by $x_2^2 - \frac{1}{4}x_1^2 x_3^2$. As g is 3-K determined, all terms of degree greater than three may be eliminated by a change of co-ordinates. Hence, up to a change of co-ordinates, we may write

$$g(x_1, \ldots, x_n) = x_1^3 \pm x_2^2 \pm \ldots \pm x_n^2.$$

Hence the germ g has an A_2 singularity at the origin. □

Corollary 4.6.4 *If $\pi : E \to B$ is a smooth fibre bundle whose base has dimension zero or one then the set of parametrized generalized Morse functions in $C^\infty(E, \mathbf{R})$ is open and dense in the Whitney C^∞ topology.*

Proof Let n in $\mathbf{N}$ be the dimension of the fibre. By Theorem A.8.8, the singular locus of a variety is an algebraic subset of strictly smaller dimension than that of the variety. Hence the union of all the irreducible components of $W_{n,1}^{n+2}$ not of the highest dimension and the singular loci of the irreducible components of highest dimension is an algebraic subset of $J_{n,1}^{n+2,0}$ of codimension greater than $n+1$. This union is equal to

$$\coprod_{i'} W_{n,1}^{n+2,i'}$$

where i' in $\{1, \ldots, i_{n+2}\}$ satisfies

$$\operatorname{codim} W_{n,1}^{n+2,i'} \geq n+2.$$

If m equals zero or one the the dimension of $E \leq n+1$. It follows that

$$\mathcal{F} = \{f \in C^\infty(E, \mathbf{R}) \ : \ j_{fibre}^{n+2} f(E) \cap (\coprod_{i'} N_{n,1}^{n+2,i'}) = \emptyset\}$$

is equal to

$$\cap_{i'} T_{N_{n,1}^{n+2,i'}}$$

where the $N_{n,1}^{n+2,i'}$ are the subbundles of $J_{fibre}^{n+2}(E, \mathbf{R})$ corresponding in local co-ordinates on fibres to the algebraic sets $W_{n,1}^{n+2,i'}$.

As the disjoint union

$$\coprod_{i'} N_{n,1}^{n+2,i'}$$

is closed in $J_{fibre}^{n+2}(E, \mathbf{R})$, by Theorem 2.2.13 and Corollary 2.2.17, $\mathcal{F}$ is residual and open. As $C^\infty(E, \mathbf{R})$ is a Baire space, $\mathcal{F}$ is dense and open. By Lemma 4.6.3, f in $C^\infty(E, \mathbf{R})$ is a parametrized generalized Morse function iff $f \in \mathcal{F}$. □

Recall our motivating question. Do all smooth bundles admit parametrized generalized Morse functions? We answer this question in part in the next chapter.

Chapter 5

A Brief Survey

In this chapter we relate the results in Chapter 4 to those in the literature. Theorem 4.3.1 is compared with similar results by Igusa [23] and Arnol'd [5] for the case $p = 1$. Complications arise due to the various ways of defining the codimension of a singularity. Those we examine coincide in the case $n = p = 1$ but differ otherwise. Igusa states his C^1 approximation in terms of a measure he calls "total codimension" (Tcod) and proves the existence of parametrized generalized Morse functions on a large class of bundles.

5.1 Igusa's C^1 Approximation for Functions on Bundles

Let $\pi : E \to B$ be a smooth fibre bundle with fibre a manifold of dimension n and base B a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Igusa [23] proves the following results.

Theorem 5.1.1 *Any smooth function $f : E \to \mathbf{R}$ may be approximated by a parametrized generalized Morse function when $m \leq n$ which is arbitrarily C^1 close to f on fibres.*

Theorem 5.1.2 *Let M be a smooth compact n-manifold and let $f : M \times D^m \to \mathbf{R}$ be a smooth function where D^m is the closed unit ball in $\mathbf{R}^m$, $1 \le n < \infty$, $0 \le m < \infty$. Let f_t be the restriction of f to $M \times \{t\}$, for every t in D^m. Then there is a smooth function $g : M \times D^m \to \mathbf{R}$ such that, on fibres, all the singularities of g are of "total codimension" $\le \max\{1, m-n+1\}$ and, on fibres, g is arbitrarily C^1 close to f.*

Corollary 5.1.3 *Any smooth function $f : E \to \mathbf{R}$ may be approximated by a smooth function g such that, on fibres, all the singularities of g are of total codimension $\le \max\{1, m-n+1\}$ and, on fibres, g is arbitrarily C^1 close to f.*

For the proof of the corollary above see Corollary 1.6, Corollary 1.7 and Corollary 2.13 in [23]. Total codimension is defined axiomatically in [23]. There it is shown that for any n in $\mathbf{N}$ there exists a total codimension function

$$\mathrm{Tcod} \ : \ M_n^2 \ \to \ \mathbf{N} \cup \{0, \infty\}$$

where $\mathbf{N}$ is the set of natural numbers.

In order to compare Theorem 4.3.1 with the results above it is necessary to relate the total codimension of a singularity to the codimension of the contact orbit of its germ. This requires defining two more measures of complexity which we will call Arnol'd's codimension and the orbital codimension, following Igusa [23].

Definition 5.1.4 *Given analytic functions $g_1, \ldots, g_n$ of n complex or real variables with an isolated common zero at $\underline{z_0}$ we define the multiplicity μ to be the degree of the associated map*

$$\underline{z} \mapsto g(\underline{z})/||g(\underline{z})||$$

from the ϵ-sphere centred at $\underline{z_0}$ to the unit sphere (see [36, page 111]).

It turns out that the multiplicity (or Milnor number) of an isolated singularity is equal to the number of nondegenerate singularities that result from a generic perturbation (or morsification) of the singularity (see [36, page 114]).

Definition 5.1.5 *Let f in M_n^2 have an isolated singularity at the origin with multiplicity $\mu(f)$. Then Acod(f) is the codimension of the stratum $\mu = \mu(f)$ in M_n^2.*

This is Arnol'd's definition of the codimension of a singularity, (see [5, page 24] and [23]).

Definition 5.1.6 *For f in M_n^2, the orbital codimension of f, denoted Ocod(f), is equal to the codimension of the ideal $M_n J(f)$ in M_n^2.*

Note that for f in M_n^2, n in $\mathbb{N}$,

$$\mathrm{Ocod}(f) = \dim M_n^2 / M_n J(f) = \dim M_n / M_n J(f) - n.$$

Recall that for f in M_n^2, n in $\mathbb{N}$,

$$\mathrm{codim}(TKf) = \dim(M_n / M_n J(f) + f^* M_1).$$

Hence, for f in M_n^2, n in $\mathbb{N}$,

$$\mathrm{Ocod}(f) + n \geq \mathrm{codim}(TKf).$$

Lemma 5.1.7 *For any n in $\mathbb{N}$ and any total codimension function*

$$Tcod \;:\; M_n^2 \;\to\; \mathbb{N} \cup \{0, \infty\}$$

and f in M_n^2,

$$Acod(f) \leq Tcod(f) \leq Ocod(f).$$

Moreover, for any integer c, $0 \leq c \leq 5$, the sets $Acod^{-1}(c)$, $Tcod^{-1}(c)$ and $Ocod^{-1}(c)$ are equal.

For proof see Igusa [23].

Corollary 5.1.8 *For any n in $\mathbb{N}$ and any total codimension function*

$$Tcod \ : \ M_n^2 \ \rightarrow \ \mathbb{N} \cup \{0, \infty\}$$

and f in M_n^2,

$$Acod(f) \leq Tcod(f) \leq Ocod(f)$$

and

$$Ocod(f) \geq codim(TKf) - n.$$

The relationship between $\text{Tcod}(f)$ and $\text{codim}(TKf)$ is not clear in general. However, by introducing the modality of a singularity we can relate the four measures of complexity in the case $n = 1$.

Definition 5.1.9 *Let f in M_n^2 have an isolated singularity at the origin with multiplicity $\mu(f)$. The modality $m(f)$ is the codimension of the orbit of f under the action of L_n in the space of all isolated singularities with the same multiplicity (see [23]).*

It turns out that (see [23]) for f in M_n^2,

$$\text{Acod}(f) + m(f) = \text{Ocod}(f) = \mu(f) - 1.$$

By Arnol'd [5, page 24], the A_k singularities have modality zero, for every k in $\mathbb{N}$. Hence, if f in M_n^2 is an A_k singularity then

$$\text{Acod}(f) = \text{Tcod}(f) = \text{Ocod}(f) = \mu(f) - 1 = \text{codim}(TKf) - n = k - 1.$$

Hence, in the case $n = 1$, f in M_1^2 is an A_k singularity iff

$$\text{Acod}(f) = \text{Tcod}(f) = \text{Ocod}(f) = \text{codim}(TKf) - 1 = k - 1.$$

The following result follows from Theorem 4.3.1.

Corollary 5.1.10 *If $n = 1$ then any element f in $C^\infty(E, \mathbf{R})$ may be arbitrarily closely C^k approximated, for any $k \geq 0$, by some g in $C^\infty(E, \mathbf{R})$ such that*

$$Acod(g) = Tcod(g) = Ocod(g) \leq m$$

everywhere on fibres.

Let us compare our Theorem 4.3.1 and Igusa's Theorem 5.1.1 and Corollary 5.1.3. Roughly speaking, for smooth real-valued functions on bundles, Igusa obtains weaker approximations by smooth functions with simpler singularities on fibres. His approximations are C^1 close on fibres. In the case $m \leq n$, these singularities need be only of type A_1 or A_2. In the case $m = 1$, Theorem 4.3.1 establishes a stronger (C^∞) approximation by parametrized generalized Morse functions. Corollary 5.1.3 proves the existence of parametrized generalized Morse functions on all smooth bundles such that $m \leq n$. It is still not known if all smooth bundles such that $m > n$ admit parametrized generalized Morse functions. Some of these bundles do, for example the Hopf bundle

$$\pi^1 \;:\; S^3 \;\rightarrow\; \mathbf{CP}^1$$

admits a parametrized generalized Morse function (see Theorem 9.3.1).

Arnol'd [5] states that for $n \geq p = 1$, "the class of codimension c is irremovable by a small perturbation only if the number of parameters is $l \geq c$. " Hence, the class of codimension c is removable by a small perturbation if the number of parameters is $l < c$. According to Arnol'd's definition of codimension, A_k singularities have codimension $k - 1$. This is essentially Theorem 4.3.1 when $n \geq p = 1$ for a different definition of codimension.

Chapter 6

Parametrized Morse Functions

In this chapter it is shown that if E is a Riemannian manifold and $\pi : E \to B$ is a smooth compact fibre bundle which has a fibre of dimension one and admits a parametrized Morse function then the parametrized Morse functions are dense in $C^\infty(E, \mathbf{R})$ in the Whitney C^0 topology (see Theorem 6.5.1).

Given any smooth real-valued function on the total space E, the idea is to approximate it by adding on a small perturbation that is suitably bumpy on fibres. By small, we mean close to zero in value and by bumpy, we mean having many critical points. The approximation will be Morse on fibres if the perturbation is bumpy enough.

It has been proved by H. Chaltin (see [23]) that the parametrized Morse functions are C^0 dense on trivial smooth compact fibre bundles when the fibre is a manifold of dimension n in $\mathbf{N}$. The proof involves constructing an arbitrarily small and bumpy Morse function from a given Morse function on a fibre and extending the small and bumpy function to the total space of the bundle so that it is the same up to diffeomorphism on all fibres. This extension is not valid on all non-trivial bundles.

However, if a smooth compact bundle has a fibre of dimension one and admits a parametrized Morse function, then it is possible to construct an arbitrarily small and bumpy Morse function on a chosen fibre and extend it to the total space. In preparation for the proof of Theorem 6.5.1, in §6.2 we show how to construct a small and bumpy Morse function on a manifold of dimension n, when $n \geq 1$, from a given Morse function (see [23]). The cases n greater than one are included to demonstrate why the proof of Theorem 6.5.1 does not generalize to bundles with fibres of dimension exceeding one. Further preparation is given in §6.3 and in §6.4. The path components of the flow of a Morse function are defined in §6.3. In §6.4 the path components of the flow of a parametrized Morse function are defined. Here we show how to "average" a given parametrized Morse function on a bundle. The average is defined when, on fibres, the gradient field of the parametrized Morse function is a Morse-Smale vector field.

In the remainder of this chapter we show that for all n-sphere bundles over the circle and all smooth bundles with fibre S^1 or S^2 over the circle, the set of parametrized Morse functions is dense in the space of smooth real-valued functions with the Whitney C^0 topology. By constructing an example on the torus, we show that this set is not, in general, C^1 dense.

6.1 Riemannian Structures

Bumpy Morse functions are characterized by having, at any point, either a large derivative or a second derivative all of whose eigenvalues are large. We require a Riemannian structure so we can measure these quantities.

Definition 6.1.1 *A Riemannian structure on a smooth manifold M (denoted by $<,>$) is a smooth choice of positive definite inner product $<,>_x$ on each tangent space T_xM, where $x \in M$, smooth in the sense that whenever Y and Z*

are smooth vector fields on M, then

$$x \mapsto < Y_x, Z_x >_x$$

is a smooth function on M.

There exists a Riemannian structure on every smooth manifold. A Riemannian manifold is a differentiable manifold together with a Riemannian structure (see [49, page 52]).

Given a Riemannian structure $<,>$ on M, the length of a tangent vector Y_x in T_xM is defined to be

$$||Y_x|| = < Y_x, Y_x >_x^{1/2}.$$

The inner product $<,>_x$ induces a natural isomorphism of T_xM with the cotangent space T_xM^*, namely

$$Y_x \mapsto \tilde{Y_x}$$

where $Y_x \in T_xM$, $\tilde{Y_x} \in T_xM^*$ and

$$\tilde{Y_x}(Z_x) = < Y_x, Z_x >_x$$

for all Z_x in T_xM. The cotangent space T_xM^* inherits an inner product via this isomorphism defined by

$$< \tilde{Y_x}, \tilde{Z_x} >_x = < Y_x, Z_x >_x$$

for all Y_x and Z_x in T_xM. Given a Riemannian structure $<,>$ on M, the length of a cotangent vector $\tilde{Y_x}$ in T_xM^* is defined to be

$$\left\|\tilde{Y_x}\right\| = < \tilde{Y_x}, \tilde{Y_x} >_x^{1/2}.$$

For proofs see [49]. Similarly, one can define the length of any tensor. The restriction of a Riemannian structure on a smooth fibre bundle to a fibre is a Riemannian structure on the fibre.

6.2 Small and Bumpy Morse Functions

Theorem 6.2.1 *Let M be a smooth n-manifold, where $n \geq 1$. Then the set of smooth Morse functions on M is an open dense subset of $C^\infty(M, \mathbf{R})$ in the Whitney C^∞ topology (see [15, Theorem 6.2, page 63]).*

A smooth Morse function on a compact smooth n-manifold has a finite set of critical points, every one of which is nondegenerate and isolated (see Lemma 6.3.9).

Proposition 6.2.2 *Let M be a smooth compact Riemannian n-manifold, where $n \geq 1$, and let $g : M \to \mathbf{R}$ be a smooth Morse function. Then, given any B_1, B_2 and ϵ in $\mathbf{R}^+$, where $\mathbf{R}^+ = \{s \in \mathbf{R} : s > 0\}$, there is a smooth Morse function*

$$h : M \to \mathbf{R}$$

such that the following three properties hold:

1. *for every x in M, either $||dh_x|| \geq B_1$ or $\left|\left|d^2 h_x(u)\right|\right| \geq B_2$ for all u in $T_x M$ such that $||u|| = 1$, where length is measured as a tensor;*
2. *for every x in M, $||h(x)|| < \epsilon$ and*
3. *each critical point of g is a critical point of h with the same index.*

Moreover, if the dimension of M is one, then level sets of g are contained in level sets of h.

The critical point set of h is likely to be much larger than the critical point set of g.

Proof This proof is based on the proof of Chaltin's theorem given in [23]. Choose a smooth Morse function

$$f \ : \ M \ \rightarrow \ \mathbf{R}$$

such that the following six conditions hold:

4. f is a small perturbation of

$$g - \min_{x \in M}\{g(x)\} \ : \ M \ \rightarrow \ \mathbf{R};$$

5. $\min_{x \in M}\{f(x)\} = 0;$

6. f and g have identical critical point sets;

7. the perturbation leaves the indices of the critical points unchanged;

8. the critical values of f are rational and

9. the level sets of g are level sets of f.

As $f \ : \ M \rightarrow \mathbf{R}$ is Morse and M is compact there exists some B_f in $\mathbf{R}^+$ such that

$$||df_x|| + \min_{u \in T_xM, \, ||u||=1}\{\left|\left|d^2 f_x(u)\right|\right|\} \geq B_f,$$

for every x in M. At each x in M, either $||df_x|| \geq B_f/2$ or $\min_{u \in T_xM, \, ||u||=1}\{\left|\left|d^2 f_x(u)\right|\right|\} \geq B_f/2$. We prove the existence of h by induction on n.

The Case $n = 1$.

Let the critical values of f be denoted by $c_1, \ldots, c_t$ where

$$0 = c_1 < c_2 < \ldots < c_t.$$

We write

$$c_i = m_i/n_i; \ 2 \leq i \leq t, \ m_i, \ n_i \in \mathbf{N}.$$

Choose l in $\mathbf{N}$ so that

10. $l > (16B_1)/(\epsilon B_f(2\pi n_2 \dots n_t))$ and

11. $l > (16B_2)/(\epsilon B_f(2\pi n_2 \dots n_t))$.

Let $a = (2\pi l n_2 \dots n_t)$, then it follows that

12. $ac_i \in 2\pi\mathbf{Z}$, $1 \leq i \leq t$ as $c_1 = 0$ and
$ac_i = am_i/n_i = (2\pi l n_2 \dots n_{i-1} m_i n_{i+1} \dots n_t)$, $2 \leq i \leq t$;

13. $a > (16B_1)/(\epsilon B_f)$ and

14. $a^2 > a > (16B_2)/(\epsilon B_f)$.

Consider

$$\begin{array}{rcccl} h_1 & : & \mathbf{R} & \rightarrow & \mathbf{R} \qquad \text{defined by} \\ & & y & \mapsto & \frac{\epsilon}{4}\sin(ay). \end{array}$$

Our choice of a ensures that the critical points of h_1 are regular values of f. Consider the function

$$\begin{array}{rcccl} h_1 \circ f & : & M & \rightarrow & \mathbf{R} \qquad \text{defined by} \\ & & x & \mapsto & \frac{\epsilon}{4}\sin(af(x)). \end{array}$$

Taking derivatives we find

$$d(h_1 \circ f)_x = (\frac{\epsilon}{4}a)(\cos(af(x)))(df_x) \tag{6.1}$$

and

$$\begin{aligned} & d^2(h_1 \circ f)_x(u) \\ = & -(\frac{\epsilon}{4}a^2)(\sin(af(x)))(df_x)(df_x(u)) + (\frac{\epsilon}{4}a)(\cos(af(x)))(d^2 f_x)(u) \end{aligned} \tag{6.2}$$

for all u in T_xM. Note that at critical points x of f, $af(x) \in 2\pi\mathbf{Z}$ (see 12), and hence the Hessian form of $h_1 \circ f$ at x is proportional to the Hessian form of f at x. Hence the Hessians have the same index.

Choose x in $(h_1 \circ f)^{-1}(0)$, then $af(x) = k\pi$, for some k in $\mathbf{Z}$, and so

$$||d(h_1 \circ f)_x|| = \frac{\epsilon}{4} a ||df_x|| \geq 4 \frac{B_1}{B_f} ||df_x||$$

and

$$\left\| d^2(h_1 \circ f)_x(u) \right\| = \frac{\epsilon}{4} a \left\| d^2 f_x(u) \right\| \geq 4 \frac{B_2}{B_f} \left\| d^2 f_x(u) \right\|$$

for all u in $T_x M$ such that $||u|| = 1$. Hence either

$$||d(h_1 \circ f)_x|| \geq 2B_1$$

or

$$\left\| d^2(h_1 \circ f)_x(u) \right\| \geq 2B_2$$

for all u in $T_x M$ such that $||u|| = 1$.

There exists some N in $\mathbf{N}$, $N \geq 8$, such that $h_1 \circ f$ obeys the first requirement of the proposition on $[-\epsilon/N, \epsilon/N]$. For if not, there exists a sequence of points $\{x_{\hat{N}} \, : \, \hat{N} \geq 8\}$ such that for every $\hat{N} \geq 8$, $x_{\hat{N}} \in (h_1 \circ f)^{-1}[-\epsilon/\hat{N}, \epsilon/\hat{N}]$,

$$\left\| d(h_1 \circ f)_{x_{\hat{N}}} \right\| < B_1$$

and

$$\left\| d^2(h_1 \circ f)_{x_{\hat{N}}}(u) \right\| < B_2$$

for some u in $T_{x_{\hat{N}}} M$ such that $||u|| = 1$. As M is compact, the sequence has a subsequence converging to say x' in M. By continuity, $(h_1 \circ f)(x') = 0$ and

$$||d(h_1 \circ f)_{x'}|| \leq B_1$$

and

$$\left\| d^2(h_1 \circ f)_{x'}(u) \right\| \leq B_2$$

for some u in $T_{x'} M$ such that $||u|| = 1$. This is a contradiction.

The function $h_1 \circ f$ may not satisfy the first requirement of the proposition on all of M. For example, if $\cos(af(x)) = 0$ then

$$d(h_1 \circ f)_x = 0$$

and

$$d^2(h_1 \circ f)_x(u) = \pm(\frac{\epsilon}{4}a^2)(df_x)(df_x(u))$$

for all u in T_xM. It may not be true that

$$\left|\left|d^2(h_1 \circ f)_x(u)\right|\right| \geq B_2$$

for all u in T_xM such that $||u|| = 1$. So we must alter $h_1 \circ f$ in the regions $f^{-1} \circ h_1^{-1}[-\frac{\epsilon}{4}, -\frac{\epsilon}{N}]$ and $f^{-1} \circ h_1^{-1}[\frac{\epsilon}{N}, \frac{\epsilon}{4}]$. Consider $\Sigma h_1 \cap f(M)$, the set of critical points of h_1 contained in the image of f. This set is finite as h_1 is Morse and $f(M)$ is compact. Let $\Sigma h_1 \cap f(M)$ be equal to $\{y_1, \ldots, y_r\}$. Then $\cos(ay_i) = 0$ and y_i is a regular value of f, for every i, $1 \leq i \leq r$. Therefore, $f^{-1}(y_i)$ is a closed zero-dimensional submanifold of M, a finite set of points, and there exists some tubular neighbourhood (see [15]) of $f^{-1}(y_i)$, diffeomorphic to $f^{-1}(y_i) \times (-\delta_i, \delta_i)$ for some small δ_i in $\mathbf{R}^+$, such that f is given, in local co-ordinates, by

$$f(x, s) = y_i + s$$

for (x, s) in $f^{-1}(y_i) \times (-\delta_i, \delta_i)$, $1 \leq i \leq r$. Choose each δ_i small enough that the tubular neighbourhoods are disjoint. Let

$$\delta = \min_{1 \leq i \leq r} \{\delta_i\}.$$

For each i, $1 \leq i \leq r$, define the function h_{y_i} on a small open neighbourhood of $f^{-1}(y_i)$ in local co-ordinates by

$$h_{y_i}(x, s) = \{ \begin{array}{ll} \frac{\epsilon}{2} - \beta_i s^2 & \text{if } \sin(ay_i) = 1 \text{ and } |s| < \delta \\ -\frac{\epsilon}{2} + \beta_i s^2 & \text{if } \sin(ay_i) = -1 \text{ and } |s| < \delta \end{array}$$

where β_i in $\mathbf{R}^+$ is large enough that

$$\min_{u \in T_xM, ||u||=1}\{\left|\left|d^2(h_{y_i})_x(u)\right|\right|\} \geq 2B_2$$

for every x in $f^{-1}(y_i)$. Restrict the domain of h_{y_i} so that $||h_{y_i}|| \geq \frac{\epsilon}{3}$ everywhere. Note that in a neighbourhood of $f^{-1}(y_i)$, level sets of f are contained in level sets of h_{y_i}. If a is large enough to ensure that the sets $f^{-1} \circ h_1^{-1}\{0\}$ and $f^{-1}(y_i)$, $1 \leq i \leq r$, are very close, then it is possible to glue together $h_1 \circ f$ and the functions h_{y_i} to obtain a smooth function $h : M \to \mathbf{R}$, equal to $h_1 \circ f$ on a neighbourhood of $f^{-1} \circ h_1^{-1}\{0\}$, equal to h_{y_i} on a neighbourhood of $f^{-1}(y_i)$, $1 \leq i \leq r$, obeying the requirements of the proposition. Roughly speaking, h will either be very steep or will have a second derivative with only large eigenvalues about $f^{-1} \circ h_1^{-1}\{0\}$, h will have a second derivative with only large eigenvalues about $f^{-1}(y_i)$, $1 \leq i \leq r$, and will be very steep in between critical points.

If a is not large enough, replace it by $l'a$, some l' in $\mathbf{N}$, $l' \geq 2$ and repeat the construction of h. If l' is chosen large enough we find that the sets $f^{-1} \circ h_1^{-1}\{0\}$ and $f^{-1}(\Sigma h_1)$ are close enough to make possible $||dh_x|| \geq B_1$ on the required regions.

The Induction Argument

Assume that $n > 1$ and that the proposition holds for each k, $1 \leq k \leq n-1$. Follow the proof for the case $n = 1$ up to the consideration of $f^{-1}(y_i)$, y_i in $\Sigma h_1 \cap f(M)$, $1 \leq i \leq r$. In this case, each $f^{-1}(y_i)$ is a closed $(n-1)$-dimensional submanifold of M, that is, a smooth compact manifold with a Riemannian structure that is the restriction of the Riemannian structure on M. There exists a tubular neighbourhood (see [15]) of $f^{-1}(y_i)$, diffeomorphic to $f^{-1}(y_i) \times (-\delta_i, \delta_i)$ for some small δ_i in $\mathbf{R}^+$, such that f is given, in local co-ordinates, by

$$f(x, s) = y_i + s$$

for all (x, s) in $f^{-1}(y_i) \times (-\delta_i, \delta_i)$, $1 \leq i \leq r$. As in the case $n = 1$, choose each δ_i small enough that the tubular neighbourhoods are disjoint. Let

$$\delta = \min_{1 \leq i \leq r} \{\delta_i\}.$$

For each i, $1 \leq i \leq r$, define the function h_{y_i} on a small open neighbourhood of $f^{-1}(y_i)$ in local co-ordinates by

$$h_{y_i}(x,s) = \{ \begin{array}{ll} \overline{h}_{y_i}(x) - \beta_i s^2 & \text{if } \sin(ay_i) = 1 \text{ and } |s| < \delta \\ -\overline{h}_{y_i}(x) + \beta_i s^2 & \text{if } \sin(ay_i) = -1 \text{ and } |s| < \delta \end{array}$$

where

$$\overline{h}_{y_i} \ : \ f^{-1}(y_i) \ \rightarrow \ (\tfrac{\epsilon}{3}, \tfrac{\epsilon}{2})$$

is a smooth Morse function such that for every x in $f^{-1}(y_i)$ either

$$\left\| d(\overline{h}_{y_i})_x \right\| \geq 2B_1$$

or

$$\min_{u \in T_x f^{-1}(y_i),\, \|u\|=1} \{ \left\| d^2(\overline{h}_{y_i})_x(u) \right\| \} \geq 2B_2$$

and where β_i in $\mathbf{R}^+$ is large enough that for every x in $f^{-1}(y_i)$ either

$$||d(h_{y_i})_x|| \geq 2B_1$$

or

$$\min_{u \in T_x M,\, \|u\|=1} \{ \left\| d^2(h_{y_i})_x(u) \right\| \} \geq 2B_2.$$

A suitable function $\overline{h}_{y_i}$ exists by the induction assumption.

The proof is finished as in the case $n = 1$ with one modification. The level sets of f cannot be contained in the level sets of h. This is due to the fact that the dimension of $f^{-1}(y_i)$ is at least one and

$$\overline{h}_{y_i} \ : \ f^{-1}(y_i) \ \rightarrow \ \mathbf{R}$$

is Morse and therefore not constant. □

Corollary 6.2.3 *Let M be a smooth compact Riemannian n-manifold, where $n \geq 1$. Then the set of smooth Morse functions on M is a dense subset of $C^\infty(M, \mathbf{R})$ in the Whitney C^0 topology.*

Proof Let $q : M \to \mathbf{R}$ be a smooth function. Choose $\epsilon > 0$. Define the smooth function

$$\begin{array}{rcll} F(q) & : & M & \to \mathbf{R} \\ & & x & \mapsto ||dq_x|| + \max\{\left\|d^2 q_x(u)\right\| : u \in T_xM, \parallel u \parallel = 1\}. \end{array} \quad \text{by}$$

Now, $F(q)$ has a maximum, say $B(q)$, on M, since M is compact and $F(q)$ is continuous.

By Proposition 6.2.2, there exists a smooth function

$$h : M \to \mathbf{R}$$

such that

1. for every x in M, either $||dh_x|| \geq 2B(q)$ or $\left\|d^2 h_x(u)\right\| \geq 2B(q)$ for all u in T_xM such that $||u|| = 1$ and
2. for every x in M, $||h(x)|| < \epsilon$.

Consider the smooth function

$$h + q : M \to \mathbf{R}.$$

If $d(h+q)_x = 0$ then $||dh_x|| = ||dq_x|| \leq B(q)$ and so here $\left\|d^2 h_x(u)\right\| \geq 2B(q)$ for all u in T_xM such that $||u|| = 1$. Hence

$$\left\|d^2(h+q)_x(u)\right\| = \left\|d^2 h_x(u) + d^2 q_x(u)\right\| \geq B(q)$$

for all u in T_xM such that $||u|| = 1$. Hence $\left\|d^2(h+q)_x\right\|$ is nonsingular. Hence $(h+q)$ is a Morse function at x. □

Corollary 6.2.3 is no surprise. It is weaker than Theorem 6.2.1 which is proved using the Thom Transversality Theorem. Proposition 6.2.2 has value in that it may be used to prove that the parametrized Morse functions are C^0 dense on smooth compact fibre bundles which have fibres of dimension one and admit a parametrized Morse function (see §6.5).

6.3 The Path Components of the Flow of a Morse Function

Let X be a smooth, compact Riemannian n-manifold, $n \geq 1$, and let p and q be critical points of a smooth Morse function $f : X \to \mathbf{R}$.

Definition 6.3.1 *Let $\{\Phi_t(x) : t \in \mathbf{R}\}$ denote the orbit of df through x in X satisfying $\Phi_0(x) = x$. The set $\{\Phi_t(x) : t \in \mathbf{R}\}$ is also called the trajectory or flow line through x. (For more details see [39, pages 10,11].)*

Definition 6.3.2 *For a in $\mathbf{R}$ denote $f^{-1}(-\infty, a]$ by X^a.*

Definition 6.3.3 *The unstable manifold of p, denoted by $W^u(p)$, is the set of all x in X such that*

$$\lim_{t\to-\infty} \Phi_t(x) = p.$$

The stable manifold of p, denoted by $W^s(p)$, is the set of all x such that

$$\lim_{t\to\infty} \Phi_t(x) = p.$$

(For more details see [39, page 79].)

Notation 6.3.4 *Denote by D^n the closed unit ball in $\mathbf{R}^n$ with the subspace topology (see [3, page 28]) where $\mathbf{R}^n$ is given the usual topology, $n \geq 1$. Denote by $(D^n)^0$ the interior of D^n with the subspace topology. We take D^0 to be a single point with empty boundary.*

Notation 6.3.5 *Let V be a subset of a topological space. Denote by ∂V the boundary of V. Denote by S^{n-1} the boundary ∂D^n, $n \geq 1$, with the subspace topology.*

Theorem 6.3.6 *If p is a critical point of f with index λ then $W^u(p)$ is diffeomorphic to $(D^{n-\lambda})^0$ and $W^s(p)$ is diffeomorphic to $(D^\lambda)^0$.*

For proof see [41] and [39, page 161].

Corollary 6.3.7 *If $\epsilon > 0$ is small enough, then*

$$f^{-1}(f(p) + \epsilon) \cap W^u(p)$$

is diffeomorphic to $S^{n-\lambda-1}$ and

$$f^{-1}(f(p) - \epsilon) \cap W^s(p)$$

is diffeomorphic to $S^{\lambda-1}$.

Definition 6.3.8 *We define an equivalence relation on the flow lines of f by $l_1 \sim l_2$ iff*

1. *l_1 and l_2 are the same critical point or*
2. *l_1 and l_2 contain no critical points and have the same limits and if l_1 and l_2 join critical point p to critical point q and if a in $\mathbf{R}$ is such that $f(p) < a < f(q)$, then $l_1 \cap f^{-1}(a)$ may be joined to $l_2 \cap f^{-1}(a)$ by a continuous path*

   $$\alpha \; : \; [0,1] \; \rightarrow \; f^{-1}(a)$$

   such that $\alpha(t)$ is an element of a flow line with limits p and q for every t in $[0,1]$.

The equivalence classes are called the path components of the flow of f.

The remainder of this section draws heavily on the ideas and proofs contained in [35, Part 1].

Lemma 6.3.9 *A smooth Morse function on a smooth compact manifold has a finite number of critical points all of which are isolated (see [35, Part 1]).*

Definition 6.3.10 *The operation of "attaching a k-cell" is defined as follows. Let V be a topological space. If $g : S^{k-1} \to V$ is a continuous map then*

$$V \cup_g (D^k)^0$$

(V with a k-cell attached by g) is obtained by first taking the topological sum (that is, disjoint union) of V and D^k, and then identifying each element of S^{k-1} with its image in V.

Theorem 6.3.11 *Let X be a smooth compact Riemannian n-manifold, $n \geq 1$ and let $f : X \to \mathbf{R}$ be a smooth Morse function. Let c in $\mathbf{R}$ be a critical value of f and let $p_1, \ldots, p_J$ be the critical points of f in the level set $f^{-1}(c)$ with indices $\lambda_1, \ldots, \lambda_J$, respectively. If ϵ in $\mathbf{R}^+$ is small enough, $X^{c+\epsilon}$ has the homotopy type of $X^{c-\epsilon}$ with a λ_j-cell attached, each λ_j, $1 \leq j \leq J$. If $\lambda_j \geq 1$, then the λ_j-cell is attached by its boundary to $X^{c-\epsilon}$. In addition, $X^{c+\epsilon}$ is diffeomorphic to $X^{c-\epsilon}$ with an n-cell attached, for every λ_j, $1 \leq j \leq J$. The n-cell is attached by the boundary $(\partial(\lambda_j\text{-cell})) \times ((n - \lambda_j)\text{-cell})$ to $X^{c-\epsilon}$ if $\lambda_j \geq 1$.*

For proof see [35].

Definition 6.3.12 *If X is a smooth compact Riemannian manifold and $f : X \to \mathbf{R}$ is a smooth Morse function then the gradient vector field of f is a Morse-Smale vector field if, for each ordered pair (p, q) of critical points of f, the unstable manifold $W^u(p)$ is transverse to the stable manifold $W^s(q)$.*

See [39, page 118] for a description of Morse-Smale vector fields.

Theorem 6.3.13 *If X is a smooth compact Riemannian n-manifold and $f : X \to \mathbf{R}$ is a smooth Morse function such that the gradient vector field of f is a Morse-Smale vector field then the number of path components of the flow of f is finite.*

Proof The path components of the flow of f correspond to

1. the critical points of f and
2. for every pair (p, q) of critical points of f such that $f(p) < f(q)$, the path components of

$$f^{-1}(\frac{1}{2}(f(p) + f(q))) \cap W^u(p) \cap W^s(q).$$

Let (p, q) be a pair of critical points of f such that $f(p) < f(q)$, let λ_p be the index of p and let λ_q be the index of q. Then the intersection

$$f^{-1}(\frac{1}{2}(f(p) + f(q))) \cap W^u(p) \cap W^s(q)$$

is diffeomorphic to a transverse intersection of the spheres $S^{n-\lambda_p-1}$ and S^{λ_q-1}, where $0 \leq \lambda_p \leq n$ and $0 \leq \lambda_q \leq n$. This intersection is a smooth submanifold of X (see [16, page 28]). Moreover, this intersection is closed in X.

Hence the intersection is a smooth compact submanifold of X. Hence the intersection has a finite number of path components. By Lemma 6.3.9, f has a finite number of critical points. Hence the number of pairs (p, q) is finite. Hence the number of path components of the flow of f is finite. □

The author conjectures that if X is a smooth compact Riemannian n-manifold and $f : X \to \mathbf{R}$ is a smooth Morse function then the number of path components of the flow of f is finite.

6.4 Averaging a Parametrized Morse Function

We study the flow on fibres of a parametrized Morse function. Let E be a Riemannian manifold and let $\pi : E \to B$ be a smooth compact fibre bundle with fibre a manifold of dimension n and base B a manifold of dimension m, $1 \leq n < \infty$ and $0 \leq m < \infty$. Let $f : E \to \mathbf{R}$ be a smooth parametrized Morse function.

Definition 6.4.1 *Let $\{\Phi_{t,b}(e) : t \in \mathbf{R}\}$ denote the orbit of df_b through e in F_b satisfying $\Phi_0(e) = e$ in F_b. The set $\{\Phi_{t,b}(e) : t \in \mathbf{R}\}$ is also called the trajectory or flow line of f on fibres through e.*

Definition 6.4.2 *We define an equivalence relation on the flow lines of f on fibres by $l_0 \sim l_1$ iff there exists a continuous path*

$$\alpha \ : \ [0,1] \ \to \ B$$

and continuous lifts of α, say τ and $\gamma : [0,1] \to \Gamma_f$, where Γ_f is the critical graph, such that the following conditions hold:

1. $\gamma(0) = \lim_{t\to-\infty} l_0$;
2. $\gamma(1) = \lim_{t\to-\infty} l_1$;
3. $\tau(0) = \lim_{t\to+\infty} l_0$;
4. $\tau(1) = \lim_{t\to+\infty} l_1$;
5. *for every s in $[0,1]$ there exists a flow line l_s such that $\gamma(s) = \lim_{t\to-\infty} l_s$ and $\tau(s) = \lim_{t\to+\infty} l_s$ and*
6. *if e_s in l_s is defined implicitly by*

$$f(e_s) = \frac{1}{2}(f(\gamma(s)) + f(\tau(s)))$$

then the map $s \mapsto e_s$ is continuous on $[0,1]$.

The equivalence classes are called the path components of the flow of f on fibres.

Definition 6.4.3 *The height function induced by f is*

$$\begin{array}{rcll} h_f : E\backslash\Gamma_f & \to & (0,1) & \text{defined by} \\ e & \mapsto & \frac{f(e)-f(e_i)}{f(e_j)-f(e_i)} & \end{array}$$

where

$$e_i = \lim_{t\to-\infty} \Phi_{t,\pi(e)}(e)$$

and

$$e_j = \lim_{t\to+\infty} \Phi_{t,\pi(e)}(e).$$

Definition 6.4.4 *We define an equivalence relation on E with respect to the parametrized Morse function f by $e_1 \sim e_2$ iff*

1. *$e_1 \in \Gamma_f$, $e_2 \in \Gamma_f$ and $e_1 \sim e_2$ as flow lines of f on fibres or*
2. *e_1 and e_2 are regular points of f on fibres, $h_f(e_1) = h_f(e_2)$ and e_1 and e_2 belong to equivalent flow lines of f on fibres.*

Denote by $[e]$ the equivalence class of e in E. We give the identification space

$$\cup_{e\in E}[e]$$

the identification topology (see [3, page 66]).

The space B is compact and so has a finite number of path components. If B is not path connected, consider its path components one by one. From now on, we assume B is path connected.

Let $[F_{b'}]$ denote the set of equivalence classes in the fibre $F_{b'}$ defined with respect to the Morse function $f_{b'}$, for every b' in B (substitute $F_{b'}$ for E in Definition 6.4.4). Each continuous path

$$\alpha \; : \; [0,1] \; \rightarrow \; B$$

such that $\alpha(0) = \alpha(1) = b$ in B defines a continuous map

$$A^{\alpha} \; : \; [F_b] \times [0,1] \; \rightarrow \; \cup_{t \in [0,1]} [F_{\alpha(t)}]$$

such that the map A^{α} preserves the heights of level sets of regular points of f on fibres and such that the map

$$\begin{array}{rcll} A^{\alpha}_t & : & [F_b] & \rightarrow & [F_{\alpha(t)}] \qquad \text{defined by} \\ & & [e] & \mapsto & A^{\alpha}([e], t) \end{array}$$

is a continuous isomorphism for every t in $[0,1]$.

If

$$\begin{array}{rcll} f^{\alpha}_b & : & F_b & \rightarrow & \mathbf{R} \qquad \text{is defined by} \\ & & e & \mapsto & f_b(A^{\alpha}_1[e]) \end{array}$$

then f^{α}_b is a smooth Morse function with the same image, flow lines and equivalence classes as has f_b.

Lemma 6.4.5 *With the assumptions above, f^{α}_b depends only on the homotopy class of α.*

Proof Suppose

$$\alpha, \beta \; : \; [0,1] \; \rightarrow \; B$$

are continuous paths such that $\alpha(0) = \beta(0) = \alpha(1) = \beta(1) = b$ for some b in B. By Proposition 2.1.3, the critical graph Γ_f is a covering space for B. Given the critical point e in $F_b \cap \Gamma_f = \Sigma f_b$, there exist unique continuous paths

$$\tilde{\alpha}, \tilde{\beta} \; : \; [0,1] \; \rightarrow \; \Gamma_f$$

such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = e$, $\pi \circ \tilde{\alpha} = \alpha$ and $\pi \circ \tilde{\beta} = \beta$. Further, if α is homotopic to β, then $\tilde{\alpha}(1)$ is equal to $\tilde{\beta}(1)$ (for proof, see [33, pages 151, 152]).

Whether or not α and β are homotopic it is clear that

$$f_b^{\beta \circ \alpha} = (f_b^{\alpha})^{\beta}.$$

Assume that α is homotopic to β and consider

$$f_b^{\beta^{-1} \circ \alpha} = (f_b^{\alpha})^{\beta^{-1}}.$$

The function $f_b^{\beta^{-1} \circ \alpha}$ equals f_b on the critical point set Σf_b, has the same flow lines and induces the same height function on $F_b \backslash \Sigma f_b$ as does f_b. Hence

$$f_b^{\beta^{-1} \circ \alpha} = (f_b^{\alpha})^{\beta^{-1}} = f_b.$$

Hence f_b^{α} is equal to f_b^{β}. □

Notation 6.4.6 *Denote by $\pi_1(B, b)$ the fundamental group of B based at b in B. From now on, write $f_b[\alpha]$ for f_b^{α} where $[\alpha] \in \pi_1(B, b)$.*

For the remainder of this section, we assume that, on fibres, the gradient vector field of $f : E \to \mathbf{R}$ is a Morse-Smale vector field.

Lemma 6.4.7 *For fixed b in B, the set*

$$S = \{f_b[\alpha] : [\alpha] \in \pi_1(B, b)\}$$

has a finite number of elements.

Proof For fixed b in B, each element in S corresponds to a permutation of the path components of the flow of the Morse function f_b. Any two elements in S corresponding to the same permutation must be equal. By Theorem 6.3.13, the number of path components of the flow of f_b is finite. □

We define an action of $\pi_1(B,b)$ on S by

$$([\beta], f_b[\alpha]) \mapsto (f_b[\alpha])[\beta] = f_b[\beta \circ \alpha].$$

Let $G_{f_b[\alpha]}$ be the isotropy subgroup of $f_b[\alpha]$ in S (see [13, page 155]). Recall that

$$[\beta] \in G_{f_b[\alpha]} \text{ iff } f_b[\beta \circ \alpha] = f_b[\alpha].$$

Let

$$G = \cap_{[\alpha]\in\pi_1(B,b)} G_{f_b[\alpha]}.$$

Then G is a subgroup of $\pi_1(B,b)$. Moreover, G is normal as G is the intersection of all conjugates of the isotropy subgroup of f_b. Explicitly, if $[\gamma] \in \pi_1(B,b)$, then

$$\begin{aligned} & [\gamma]G[\gamma]^{-1} \\ = \ & \cap_{[\alpha]\in\pi_1(B,b)}[\gamma]G_{f_b[\alpha]}[\gamma]^{-1} \\ = \ & \cap_{[\alpha]\in\pi_1(B,b)}G_{f_b[\gamma\circ\alpha]} \\ = \ & G. \end{aligned}$$

Hence the coset space $\pi_1(B,b)/G$ is a group.

Lemma 6.4.8 *The coset space $\pi_1(B,b)/G$ has a finite number of elements.*

Proof Let P be the number of elements in S and let

$$\{f_b[\alpha_1], \ldots, f_b[\alpha_P]\}$$

be a set of distinct elements of S. Define a map from $\pi_1(B,b)/G$ into the finite set of P-tuples of elements in S by

$$[\beta] \mapsto \begin{pmatrix} f_b[\beta\circ\alpha_1] \\ \vdots \\ f_b[\beta\circ\alpha_P] \end{pmatrix}.$$

The map is one-to-one as by definition $\pi_1(B,b)/G$ is effective with respect to the action on S. Hence the coset space $\pi_1(B,b)/G$ has a finite number of elements. □

Let N be the number of elements of $\pi_1(B,b)/G$ and define $\tilde{f}_b$ to be

$$\frac{1}{N}\left(\Sigma_{[\alpha]\in\pi_1(B,b)/G} f_b[\alpha]\right),$$

an average of the functions in the set S.

Lemma 6.4.9 *The map $\tilde{f}_b : F_b \to \mathbf{R}$ is smooth, Morse and has the same critical point set, flow lines, height function and equivalence classes as does $f_b[\beta]$ for any $[\beta]$ in $\pi_1(B,b)/G$. Moreover, $\tilde{f}_b$ is constant on equivalence classes defined with respect to the parametrized Morse function $f : E \to \mathbf{R}$ (in F_b).*

Proof The first statement of the lemma follows from the definition of $\tilde{f}_b$. To prove the second, choose $[\beta]$ in $\pi_1(B,b)/G$, then

$$\begin{aligned} & \tilde{f}_b[\beta](e) \\ = & \frac{1}{N}\left(\Sigma_{[\alpha]\in\pi_1(B,b)/G} f_b[\beta\circ\alpha](e)\right) \\ = & \tilde{f}_b(e) \end{aligned}$$

for all e in F_b. □

Notation 6.4.10 *We call $\tilde{f}_b$ the average of the parametrized Morse function f on F_b.*

Definition 6.4.11 *Define*

$$\begin{array}{rrcll} \tilde{f}_b^E : & E & \to & \mathbf{R} & \quad by \\ & e & \mapsto & \tilde{f}_b[e]. & \end{array}$$

For any b in B, $\tilde{f}_b^E$ is a smooth parametrized Morse function on E, with the same flow lines and critical points on fibres as has f and is constant on the equivalence classes of E defined with respect to f. In the next section we make use of the symmetry of the parametrized Morse function $\tilde{f}_b^E$.

Remember that $\tilde{f}_b^E$ is defined only when, on fibres, the gradient vector field of $f : E \to \mathbf{R}$ is a Morse-Smale vector field. Observe that, if the fibre of $\pi : E \to B$ has dimension one, then on fibres, the gradient vector field of f is a Morse-Smale vector field.

6.5 Bundles With Fibre of Dimension One

The main result of the first five sections of this chapter is the following theorem.

Theorem 6.5.1 *Let E be a Riemannian manifold and let $\pi : E \to B$ be a smooth compact fibre bundle which has a fibre of dimension one and admits a parametrized Morse function. Then the parametrized Morse functions on E are dense in $C^\infty(E, \mathbf{R})$ in the Whitney C^0 topology.*

Theorem 6.5.1 follows from Theorem 6.5.2 as Corollary 6.2.3 follows from Proposition 6.2.2.

Theorem 6.5.2 *Let E be a Riemannian manifold and let $\pi : E \to B$ be a smooth compact fibre bundle which has a fibre of dimension one and admits a parametrized Morse function. Given any B_1, B_2 and ϵ in $\mathbf{R}^+$, there exists a smooth parametrized Morse function*

$$h : E \to \mathbf{R}$$

such that on each fibre F_b, where $b \in B$, with the restriction of the Riemannian structure on E, the following properties hold for every e in F_b:

1. *either $||dh_e|| \geq B_1$ or $\left|\left|d^2 h_e(u)\right|\right| \geq B_2$, for all u in $(T_\pi E)_e$ such that $||u|| = 1$ and*
2. *$||h(e)|| < \epsilon$.*

Proof Let $f : E \to \mathbf{R}$ be a smooth parametrized Morse function. Since the fibre of $\pi : E \to B$ has dimension one, it follows that, on fibres, the gradient vector field of f is a Morse-Smale vector field. Choose b in B and let

$$\tilde{f}_b^E \; : \; E \; \to \; \mathbf{R}$$

be the smooth parametrized Morse function defined in the previous section. Recall $\tilde{f}_b^E$ is constant on the equivalence classes of E defined with respect to parametrized Morse function f.

By Proposition 6.2.2, there exists a smooth Morse function

$$h_b \; : \; F_b \; \to \; \mathbf{R}$$

such that the following three properties hold:

1. for every e in F_b either $||d(h_b)_e|| \geq B_1$ or $\left|\left|d^2(h_b)_e(u)\right|\right| \geq B_2$, for all u in T_eF_b such that $||u|| = 1$;
2. for every e in F_b, $||h_b(e)|| < \epsilon$ and
3. h_b is constant on the equivalence classes of E defined with respect to the parametrized Morse function f (in F_b).

Define the function

$$h \; : \; E \; \to \; \mathbf{R}$$

to be equal to h_b on F_b and constant on each equivalence class of E defined with respect to the parametrized Morse function f. As the Riemannian structure varies from fibre to fibre it may not be true that h satisfies the first property of the theorem everywhere. However E is compact and h is a parametrized Morse function so the smooth real-valued function on E defined by

$$e \mapsto ||dh_e|| + \min\{\left|\left|d^2(h_{\pi(e)})_e(u)\right|\right| \; : \; u \in T_eF_{\pi(e)}, \;\; ||u|| = 1\}$$

has a minimum greater than zero. Hence if h_b is chosen bumpy enough, h will satisfy all the requirements of the theorem. □

6.6 Bundles Over the Circle

In this section we prove that if E is a Riemannian manifold and $\pi : E \to S^1$ is a smooth compact n-sphere bundle, $1 \leq n < \infty$, then the parametrized Morse functions are dense in $C^\infty(E, \mathbf{R})$ in the Whitney C^0 topology. The same conclusion holds for all smooth fibre bundles over the circle with fibre S^1 or S^2.

Let E be a Riemannian manifold and let $\pi : E \to S^1$ be a smooth compact fibre bundle which has a fibre X of dimension n, $1 \leq n < \infty$, and a group we call G. Consider the total space E as the product space $X \times [0,1]$ with $(X,1)$ identified with $(X,0)$ by some g in G and write

$$\begin{aligned} g \ : \ (X,1) &\rightarrow (X,0) \\ (x,1) &\mapsto (g(x),0). \end{aligned}$$

Definition 6.6.1 *An isotopy is a homotopy*

$$H \ : \ X \times [0,1] \ \rightarrow \ X$$

such that each map H_t, t in $[0,1]$, is a diffeomorphism. Two diffeomorphisms are isotopic if they can be joined by an isotopy.

Proposition 6.6.2 *The bundle $\pi : E \to S^1$ admits a parametrized Morse function if there exists a smooth Morse function $h : X \to \mathbf{R}$ and a diffeomorphism $g' : X \to X$, g' in G, such that*

1. *g' is isotopic to g and*
2. *$h \circ g' = h$.*

Proof Let $H : X \times [0,1] \to X$ be an isotopy joining g' to g. Then $(g')^{-1} \circ H$ is an isotopy joining the identity to $(g')^{-1} \circ g$.

Define the function

$$\begin{array}{rcrcl} F & : & X \times [0,1] & \to & \mathbf{R} \\ & & (x,t) & \mapsto & h \circ (g')^{-1} \circ H_t(x). \end{array} \qquad \text{by}$$

Let $F_t(x)$ equal $F(x,t)$ for every x in X and every t in $[0,1]$. Then $F_0 = h$, F_t is Morse for all t in $[0,1]$ and $F_1 = h \circ (g')^{-1} \circ g = h \circ g$. The function h induces a continuous function on E which is Morse and smooth on fibres. By Corollary 6.6.14, there exists a smooth function on E which is Morse on fibres. $\square$

Definition 6.6.3 *Any bundle in which the fibre is an n-sphere and the group is the orthogonal group O_{n+1} is called an n-sphere bundle, $n \geq 1$ (see [44, page 34]).*

The orthogonal group O_{n+1} has two connected components

1. the subgroup of matrices with determinant equal to plus one and
2. a second component of matrices with determinant equal to minus one.

An n-sphere bundle over S^1 is equivalent to

1. the product bundle if $\det(g) = +1$ or
2. a "generalized Klein bottle" if $\det(g) = -1$.

The generalized Klein bottle is constructed by forming the product of S^n with the interval $[0,1]$ and matching the ends with an orientation reversing transformation. It is non-orientable and hence not equivalent to a product bundle.

Proposition 6.6.4 *All n-sphere bundles over S^1 admit a parametrized Morse function, which, on fibres, is diffeomorphic to the height function.*

Proof If $\det(g) = 1$, then g is isotopic to the identity. In the proof of Proposition 6.6.2, take g' to be the identity and h to be the height function

$$\begin{array}{rccl} height : & S^n & \to & \mathbb{R} \quad \text{defined by} \\ & \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} & \mapsto & x_1. \end{array}$$

Then there is a continuous real-valued function on E that is diffeomorphic to the height function on fibres. By Corollary 6.6.13, there exists a smooth real-valued function on E that is diffeomorphic to the height function on fibres.

If $\det(g) = -1$, then g is isotopic to the orientation reversing reflection

$$\begin{array}{rccc} r : & S^n & \to & S^n \qquad \text{defined by} \\ & \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} & \mapsto & \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ -x_{n+1} \end{pmatrix}. \end{array}$$

The height function is Morse and invariant under r. In the proof of Proposition 6.6.2, take g' to be r and h to be the height function. Then there is a continuous real-valued function on E that is diffeomorphic to the height function on fibres. By Corollary 6.6.13, there exists a smooth real-valued function on E that is diffeomorphic to the height function on fibres. $\square$

Is it true that, for any n in $\mathbb{N}$, all bundles over S^1 with fibre S^n admit a parametrized Morse function? To answer this question we need to know more about $\mathrm{Diff}(S^n)$.

Lemma 6.6.5 *The relation "f is isotopic to g" is an equivalence relation on $\mathit{Diff}(X)$.*

For proof see [21, page 111].

Lemma 6.6.6 *1. Diff(S^1) has exactly two isotopy classes.*

2. Diff(S^2) has exactly two isotopy classes.

For proof see [21, page 186] and [38], respectively.

Corollary 6.6.7 *Any orientation preserving, respectively reversing, g in Diff(S^n) is isotopic to the identity, respectively r, the orientation reversing reflection, if n is one or two.*

Corollary 6.6.8 *Any fibre bundle over S^1 with fibre S^1 or S^2 admits a parametrized Morse function which, on fibres, is diffeomorphic to the height function.*

Theorem 6.6.9 The Hopf Degree Theorem
Two maps of a compact connected oriented k-manifold : $X \to S^k$ are homotopic iff they have the same degree, when $k \geq 1$.

A definition of the degree of a map and the proof of Theorem 6.6.9 is in [16]. The degree of g in Diff(S^n) is plus one iff g preserves orientation. The degree of g in Diff(S^n) is minus one iff g reverses orientation. Hence, for any n in $\mathbb{N}$, g in Diff(S^n) is homotopic to the identity or to r. In general it is not true that g in Diff(S^n) is isotopic to the identity or to r when $n \geq 3$. There are at least 28 isotopy classes with degree plus one in Diff(S^6) (see [21, page 187]). Hence we cannot be sure that all bundles over S^1 with fibre S^6 admit a parametrized Morse function.

Theorem 6.6.10 *Let E be a Riemannian manifold and let $\pi : E \to S^1$ be a smooth compact n-sphere bundle, $1 \leq n < \infty$. Then the parametrized Morse functions are dense in $C^\infty(E, \mathbf{R})$ in the Whitney C^0 topology.*

Theorem 6.6.10 follows from Theorem 6.6.11 as Corollary 6.2.3 follows from Proposition 6.2.2.

Theorem 6.6.11 *Let E be a Riemannian manifold and let $\pi : E \to S^1$ be a smooth compact n-sphere bundle, $1 \leq n < \infty$. Given any B_1, B_2, and ϵ in $\mathbf{R}^+$ there exists a smooth parametrized Morse function*

$$h : E \to \mathbf{R}$$

such that on each fibre F_b, where $b \in B$, with the restriction of the Riemannian structure on E, the following two properties hold:

1. *for every e in F_b, either $||dh_e|| \geq B_1$ or $\left\|d^2 h_e(u)\right\| \geq B_2$ for all u in $(T_\pi E)_e$ such that $||u|| = 1$ and*
2. *for every e in F_b, $||h(e)|| < \epsilon$.*

Proof The theorem is proved by induction. By Theorem 6.5.1 and Proposition 6.6.4 the theorem is true for n equal to one. Assume the theorem is true for some n in $\mathbf{N}$. Let $\pi : E \to S^1$ be a smooth compact Riemannian $(n+1)$-sphere bundle. By Proposition 6.6.4 there exists a smooth parametrized Morse function

$$f : E \to \mathbf{R}$$

such that, up to diffeomorphism, f is the height function on each fibre. The gradient vector field of the height function on a sphere is a Morse-Smale vector field. Average the parametrized Morse function f and adjust the values so that on each fibre the average $\tilde{f}$ has a minimum of zero and a maximum of one.

Each level set of regular points (with respect to $\tilde{f}$ on fibres) is the total space of an n-sphere bundle over S^1. To justify this statement in the case that E is not orientable, identify $\mathbf{R}^n \subset \mathbf{R}^{n+1}$ with the tangent space at a point in a level set of regular points. As the height function measures a co-ordinate which is independent of the reflected co-ordinate we may take $O_{n+1} \subset O_{n+2}$ to be the group of the bundle with total space equal to the level set of regular points. By the induction assumption, the parametrized Morse functions are C^0 dense on each level set of regular points.

Using the notation of Proposition 6.2.2, consider the function

$$h_1 \circ \tilde{f} \ : \ E \ \rightarrow \ \mathbf{R}$$

restricted to the fibres. Suppose $y_i \in \Sigma h_1 \cap \tilde{f}(E)$, $1 \leq i \leq r$. Then $\tilde{f}^{-1}(y_i)$ is a level set of regular points in E, that is, the total space of an n-sphere bundle over S^1, $1 \leq i \leq r$. By the induction assumption, there exists a smooth parametrized Morse function

$$\overline{h}_{y_i} \ : \ \tilde{f}^{-1}(y_i) \ \rightarrow \ \mathbf{R}$$

such that for every e in $\tilde{f}^{-1}(y_i)$, $\frac{\epsilon}{3} \leq \overline{h}_{y_i}(e) \leq \frac{\epsilon}{2}$, and on fibres either $\left\|(d\overline{h}_{y_i})_e\right\| \geq 2B_1$ or $\left\|(d^2\overline{h}_{y_i})_e(u)\right\| \geq 2B_2$ for all u in $(T_\pi \tilde{f}^{-1}(y_i))_e$ such that $||u|| = 1$. For each i, $1 \leq i \leq r$, define the function h_{y_i} on a small open neighbourhood of $\tilde{f}^{-1}(y_i)$ in local co-ordinates by

$$h_{y_i}(e,s) = \left\{ \begin{array}{ll} \overline{h}_{y_i}(e) - \beta_i s^2 & \text{if } \sin(ay_i) = 1 \\ -\overline{h}_{y_i}(e) + \beta_i s^2 & \text{if } \sin(ay_i) = -1 \end{array} \right.$$

where β_i in $\mathbf{R}^+$ is large enough that the absolute value of each eigenvalue of the second derivative is at least B_2 near e, for every e in $\tilde{f}^{-1}(y_i)$, on fibres. As in the proof of Proposition 6.2.2, glue together the functions $h_1 \circ \tilde{f}$ and h_{y_i}, $1 \leq i \leq r$, to obtain a smooth function $h : E \rightarrow \mathbf{R}$, equal to $h_1 \circ \tilde{f}$ on a neighbourhood of $\tilde{f}^{-1} \circ h_1^{-1}\{0\}$, equal to h_{y_i} on a neighbourhood of $\tilde{f}^{-1}(y_i)$, $1 \leq i \leq r$, satisfying the two properties of the theorem.

If a (see Proposition 6.2.2) is not large enough to ensure that the sets $\tilde{f}^{-1} \circ h_1^{-1}\{0\}$ and $\tilde{f}^{-1}(y_i)$ are close enough, then replace it by $l'a$, some l' in $\mathbf{N}$, $l' \geq 2$, and repeat the construction of h. If l' is chosen large enough we find the sets $\tilde{f}^{-1} \circ h_1^{-1}\{0\}$ and $\tilde{f}^{-1}(\Sigma h_1)$ are close enough to make possible $||dh_e|| \geq B_1$ on the required regions, on fibres. □

The same proof may be used to show that if $\pi : E \to S^1$ is a smooth fibre bundle with fibre S^1 or S^2 then the parametrized Morse functions are dense in $C^\infty(E, \mathbf{R})$ in the Whitney C^0 topology.

The following smoothing lemma ia adapted from [37, Lemma 4.1]. It will be used to prove Corollary 6.6.14 which was needed for the proof of Proposition 6.6.2.

Lemma 6.6.12 *Let U be an open subset of $\mathbf{R}$. Let A be a compact subset of the open set $V \subset \mathbf{R}$ such that the closure $\overline{V}$ is contained in U, and $\overline{V}$ is compact. Let X be a smooth compact Riemannian n-manifold, $n \geq 1$. Let $f : U \times X \to \mathbf{R}$ be continuous on $U \times X$ and smooth on each fibre $\{t\} \times X$, t in U. Let δ be a positive number. Let $f_t(x)$ equal $f(t,x)$ where $x \in X$. Then there is a continuous map $\tilde{f} : U \times X \to \mathbf{R}$ such that*

1. *$\tilde{f}$ is of class C^∞ on a neighbourhood of $A \times X$,*
2. *$\tilde{f}$ equals f outside $V \times X$,*
3. *for every t in U and every x in X*
$$\left|\tilde{f}_t(x) - f_t(x)\right| < \delta,$$
4. *for every t in U and every x in X*
$$\left\|d\tilde{f}_t(x) - df_t(x)\right\| < \delta$$
and

5. *for every t in U and every x in X and for every u in T_xX such that* $||u|| = 1$

$$\left\|d^2\tilde{f_{t,x}}(u) - d^2 f_{t,x}(u)\right\| < \delta.$$

Proof Let W be an open set containing A such that $\overline{W} \subset V$. Let Ψ be a C^∞ function on $\mathbf{R}$ which equals 1 in a neighbourhood of A and equals zero outside W. Define

$$\begin{array}{rccll} g & : & U \times X & \rightarrow & \mathbf{R} \qquad\qquad \text{by} \\ & & (t,x) & \mapsto & \Psi(t) \cdot f(t,x). \end{array}$$

Extend g to $\mathbf{R} \times X$ by letting it equal zero outside $\overline{W} \times X$, then g is continuous everywhere and smooth on fibres.

Let $\Phi(t)$ be a smooth function on $\mathbf{R}$ which is positive on the interior of the closed interval $[-\epsilon, \epsilon]$ and zero elsewhere. Here ϵ is yet to be chosen. Assume that

$$\int_{[-\epsilon,\epsilon]} \Phi(t)dt = 1.$$

Define the function

$$\begin{array}{rccll} h & : & U \times X & \rightarrow & \mathbf{R} \qquad\qquad \text{by} \\ & & (t,x) & \mapsto & \int_{[-\epsilon,\epsilon]} \Phi(s)g(t+s,x)ds. \end{array}$$

Choose ϵ less than the distance from W to the complement of V. Then $h(t,x) = 0$ for t outside V.

Define the function

$$\begin{array}{rccll} \tilde{f} & : & U \times X & \rightarrow & \mathbf{R} \qquad\qquad \text{by} \\ & & (t,x) & \mapsto & f(t,x)(1-\Psi(t)) + h(t,x). \end{array}$$

Since $\Psi(t) = 0$ and $h(t,x) = 0$ for t outside V the second requirement of the lemma is satisfied.

Now

$$h(t,x) \;=\; \int_{[-\epsilon,\epsilon]} \Phi(s)g(t+s,x)ds$$

$$
\begin{aligned}
&= \int_{t+[-\epsilon,\epsilon]} \Phi(r-t)g(r,x)dr \\
&= \int_{\mathbb{R}} \Phi(r-t)g(r,x)dr.
\end{aligned}
$$

The function h is smooth on $U \times X$ as Φ is smooth with respect to t and g is smooth with respect to x. Thus the first requirement of the lemma is satisfied. Since $\tilde{f} = f(1-\Psi) + h$ and Ψ and h are smooth, the class of $\tilde{f}$ on any open set is no less than the class of f. Hence $\tilde{f}$ is continuous everywhere.

Now

$$\tilde{f}(t,x) = f(t,x) + h(t,x) - g(t,x).$$

To satisfy the last three requirements we need only choose ϵ small enough so that, on $U \times X$, h and g and their first and second derivatives are δ-close everywhere on fibres. By a mean value theorem, given t in U and x in X, there exists some t_0 in $[-\epsilon, \epsilon]$ such that

$$h(t,x) = g(t+t_0, x),$$

some t_1 in $[-\epsilon, \epsilon]$ such that

$$
\begin{aligned}
dh_t(x) &= \int_{\mathbb{R}} \Phi(r-t)dg_r(x)dr \\
&= dg_{t+t_1}(x)
\end{aligned}
$$

and some t_2 in $[-\epsilon, \epsilon]$ such that, given u in T_xX such that $||u|| = 1$,

$$
\begin{aligned}
d^2h_{t,x}(u) &= \int_{\mathbb{R}} \Phi(r-t)d^2g_{r,x}(u)dr \\
&= d^2g_{t+t_2,x}(u)
\end{aligned}
$$

where $h_t(x) = h(t,x)$ and $g_t(x) = g(t,x)$.

The functions g, dg_t and d^2g_t are uniformly continuous on compact subspaces of their respective domains. Choose ϵ small enough that, for any chosen pair (t,x),

$$|g(t,x) - g(t^*,x)| < \delta$$

and

$$||dg_t(x) - dg_{t^*}(x)|| < \delta$$

and, for any chosen triple (t, x, u) where $u \in T_xX$ and $||u|| = 1$

$$\left\|d^2 g_{t,x}(u) - d^2 g_{t^*,x}(u)\right\| < \delta$$

if t and t^* are in V and $|t - t^*| < \epsilon$. Then the last three requirements are satisfied. □

Corollary 6.6.13 *Let E be a Riemannian manifold and let $\pi : E \to S^1$ be a smooth compact fibre bundle which has a fibre X of dimension n, $1 \leq n < \infty$. Let $f : E \to \mathbf{R}$ be continuous and smooth on fibres. Then there exists some smooth real-valued function on E which approximates f on fibres as closely as required in the Whitney C^2 topology.*

Proof We construct a suitable smooth function inductively. Let the maps

$$h_i : U_i \times X \to E$$

determine smooth local trivialisations of the bundle $\pi : E \to S^1$ such that

$$E = \cup_{i=1}^2 h_i(A_i \times X),$$

the U_i are open sets in $\mathbf{R}$ and where A_i, V_i and W_i are chosen as in the previous smoothing lemma.

Let f_0 equal f. Assume that $f_{i-1} : E \to \mathbf{R}$ is a continuous function, smooth on fibres and smooth on

$$\bigcup_{j=1}^{i-1} h_j(W_j \times X)$$

for some i, $1 \leq i \leq 2$. Let

$$g_{i-1} = f_{i-1} \circ h_i : U_i \times X \to \mathbf{R}.$$

Choose $\delta > 0$ and apply the previous smoothing lemma to obtain a continuous function

$$g_i : U_i \times X \to \mathbf{R}$$

such that

1. g_i is of class C^∞ on a neighbourhood of $A_i \times X$,
2. g_i equals g_{i-1} outside $V_i \times X$,
3. g_i is $\frac{\delta}{2^i}$ close to g_{i-1} everywhere,
4. the first derivatives of g_i and g_{i-1} on fibres are $\frac{\delta}{2^i}$ close everywhere
5. the second derivatives of g_i and g_{i-1} on fibres are $\frac{\delta}{2^i}$ close everywhere, that is,

   $$\left\| d^2 g_{i,t,x}(u) - d^2 g_{i-1,t,x}(u) \right\| < \frac{\delta}{2^i}$$

   everywhere, where u in $T_x X$ is such that $||u|| = 1$ and
6. g_i is smooth on fibres.

Then f_i is well defined by the equations

1. $f_i = f_{i-1}$ outside $h_i(U_i \times X)$,
2. $f_i \circ h_i(t, x) = f_{i-1} \circ h_i(t, x)$ for t outside V_i and
3. $f_i \circ h_i(t, x) = g_i(t, x)$ for t in U_i.

The function f_i is smooth on $\bigcup_{j=1}^{i} h_j(W_j \times X)$. By induction, there exists a smooth function $f_2 : E \to \mathbf{R}$ such that f_2 is δ-close to f on fibres in the Whitney C^2 topology. □

Corollary 6.6.14 *Let E be a Riemannian manifold and let $\pi : E \to S^1$ be a smooth compact fibre bundle which has a fibre X of dimension n, $1 \le n < \infty$. If there exists a continuous real-valued function on E which is smooth and Morse on fibres then there exists a smooth real-valued function on E which is Morse on fibres.*

Corollary 6.6.14 is a direct consequence of Corollary 6.6.13.

6.7 Parametrized Morse Functions on the Torus

We give an example of a parametrized generalized Morse function mapping $S^1 \times S^1$ into $\mathbf{R}$ which cannot be arbitrarily closely C^1 approximated by a smooth parametrized Morse function.

Let the function

$$\begin{array}{rccl} f: & \mathbf{R}^2 & \to & \mathbf{R} \\ & (x,y) & \mapsto & (\cos^2\frac{y}{2})\frac{9}{7}\sin(x) + (\sin^2\frac{y}{2})(-\sin 3x). \end{array}$$

be defined by

The function f is invariant under the transformations

$$\begin{array}{rcl} (x,y) & \mapsto & (x, y+2\pi) \quad \text{and} \\ (x,y) & \mapsto & (x+2\pi, y) \end{array}$$

and so defines a function on the torus. Let the second co-ordinate denote the fibre. Calculating derivatives we find

$$\frac{\partial f(x,y)}{\partial x} = (\cos^2\frac{y}{2})\frac{9}{7}\cos(x) + (\sin^2\frac{y}{2})(-3\cos 3x), \tag{6.3}$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = (\cos^2\frac{y}{2})\frac{-9}{7}\sin(x) + (\sin^2\frac{y}{2})(9\sin 3x) \text{ and} \tag{6.4}$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = (\cos^2\frac{y}{2})\frac{-9}{7}\cos(x) + (\sin^2\frac{y}{2})(27\cos 3x). \tag{6.5}$$

Let $f_y(x)$ be equal to $f(x,y)$ for all x and y. The function f_y has a degenerate critical point at x when $\frac{\partial f(x,y)}{\partial x} = \frac{\partial^2 f(x,y)}{\partial x^2} = 0$, that is, when the matrix equation

$$\begin{pmatrix} \frac{9}{7}\cos(x) & -3\cos 3x \\ -\frac{9}{7}\sin(x) & 9\sin 3x \end{pmatrix} \begin{pmatrix} \cos^2\frac{y}{2} \\ \sin^2\frac{y}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.6}$$

is satisfied. Hence, at degenerate critical points on fibres, the 2×2 matrix above is singular and so

$$\begin{array}{rrcl} & 3\sin 3x\cos x - \sin x\cos 3x & = & 0 \\ \Rightarrow & \frac{3}{2}(\sin 4x + \sin 2x) - \frac{1}{2}(\sin 4x - \sin 2x) & = & 0 \\ \Rightarrow & \sin 4x + 2\sin 2x & = & 0 \\ \Rightarrow & \sin 2x(\cos 2x + 1) & = & 0 \\ \Rightarrow & x & = & k\pi/2, \text{ for some } k \text{ in } \mathbf{Z}. \end{array}$$

We conclude that degenerate critical points on fibres, if they exist, are of the form (x, y) where $x = k\pi/2$ and $k \in \mathbf{Z}$.

Substituting these values into equation (6.6) we find that degenerate critical points occur only at points (x, y) where $x = k\pi$, $k \in \mathbf{Z}$ and $\cos^2\frac{y}{2} = \frac{7}{10}$. Substituting into equation (6.5) we find that these degenerate critical points are A_2 singularities. Hence there exist just two fibres on which f is not Morse and on these fibres f is a generalized Morse function with just two A_2 singularities.

The points in the torus at which the first two derivatives of f vanish on fibres are illustrated in Figure 6.1. The values c, b and a marked on the horizontal y-axis are such that $\cos^2\frac{c}{2} = \frac{1}{2}$, $\cos^2\frac{b}{2} = \frac{7}{10}$ and $\cos^2\frac{a}{2} = \frac{9}{10}$. The graphs of f_y, for y equal to c, b and a, are illustrated in Figures 6.2, 6.3 and 6.4, respectively.

A very fine C^1 approximation of f on fibres will have all its critical points on fibres very close to those of f. It is clear from Figure 6.1 that no such approximation can have a critical graph that is a covering space of S^1. By Proposition 2.1.3, any very fine C^1 approximation of f on fibres is not a parametrized Morse function. Hence the parametrized Morse functions are not dense in $C^\infty(S^1 \times S^1, \mathbf{R})$ in the Whitney C^1 topology.

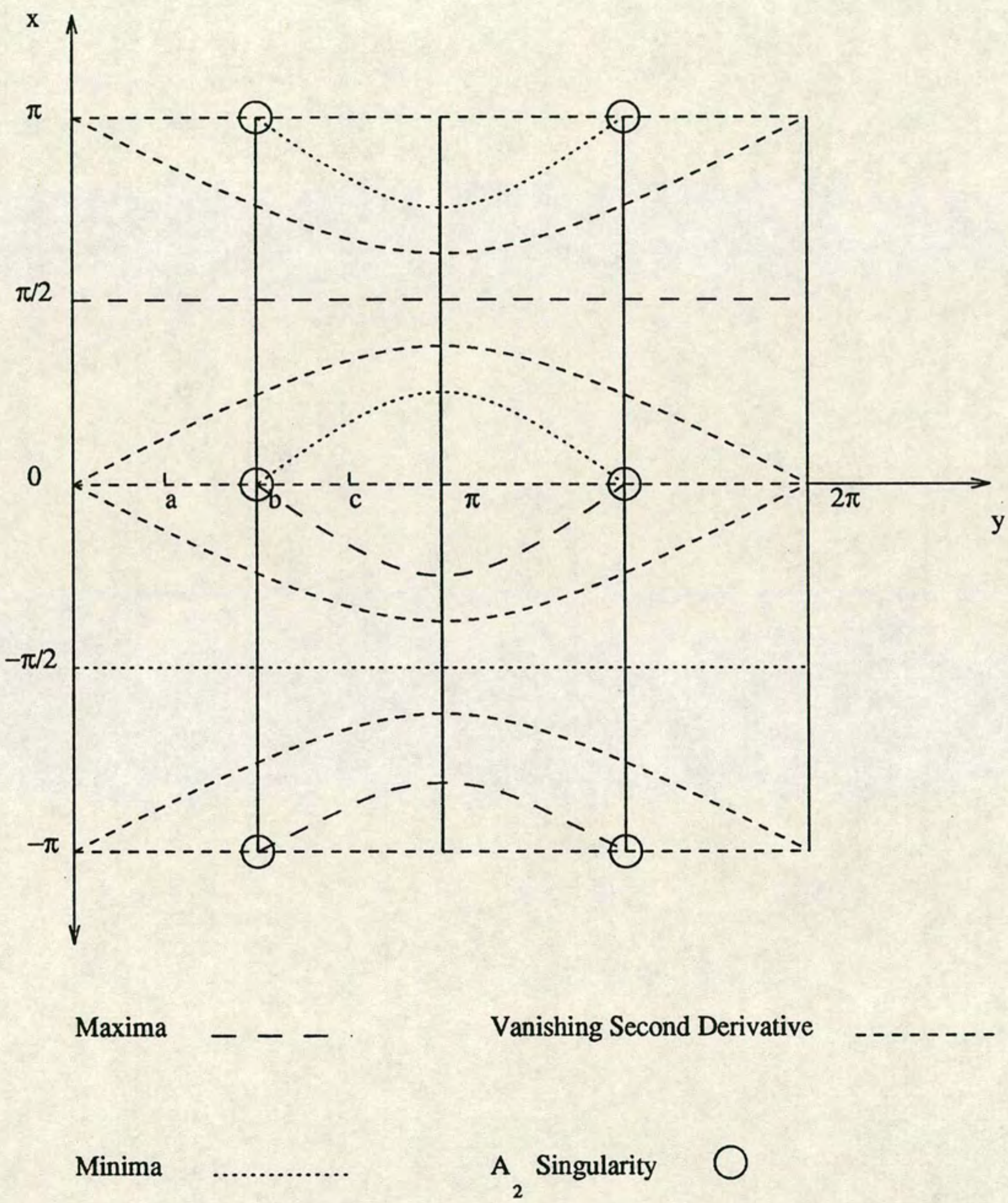

Figure 6.1: Vanishing Derivatives on Fibres of f on the Torus

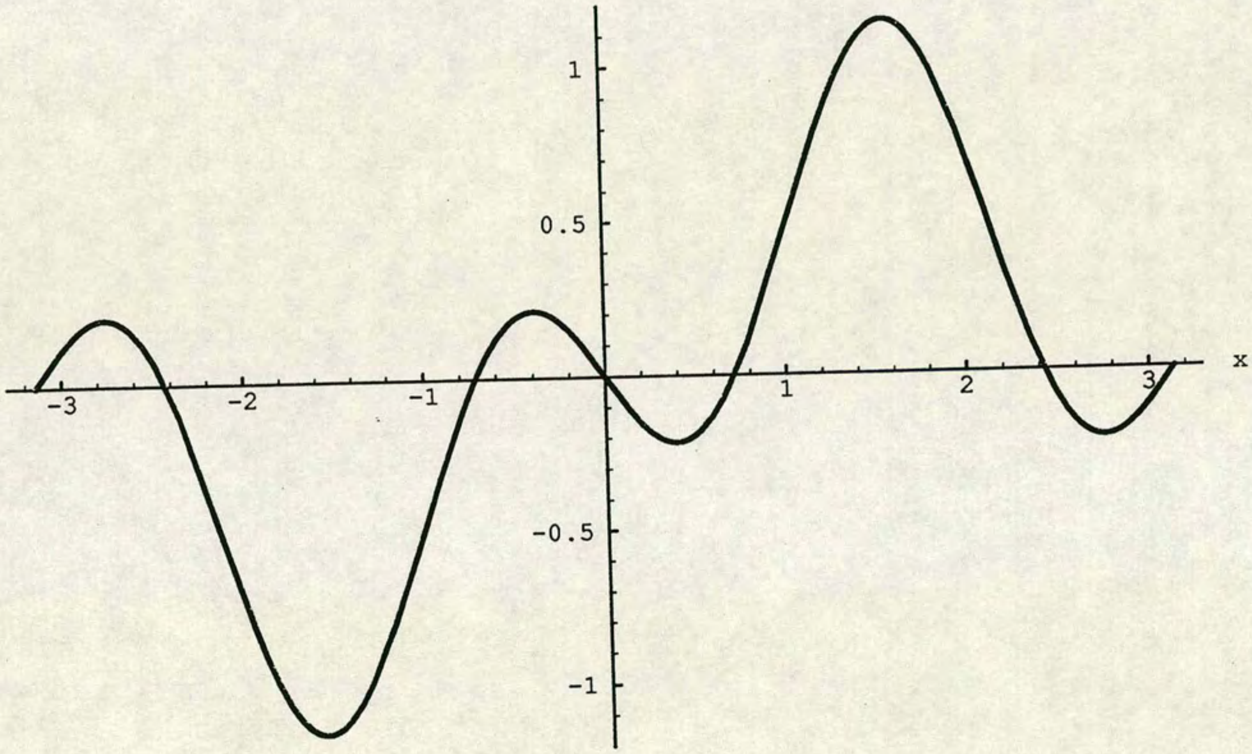

Figure 6.2: $f_y(x) = \frac{9}{14}\sin(x) - \frac{1}{2}\sin(3x)$, $\cos^2\frac{y}{2} = \frac{1}{2}$

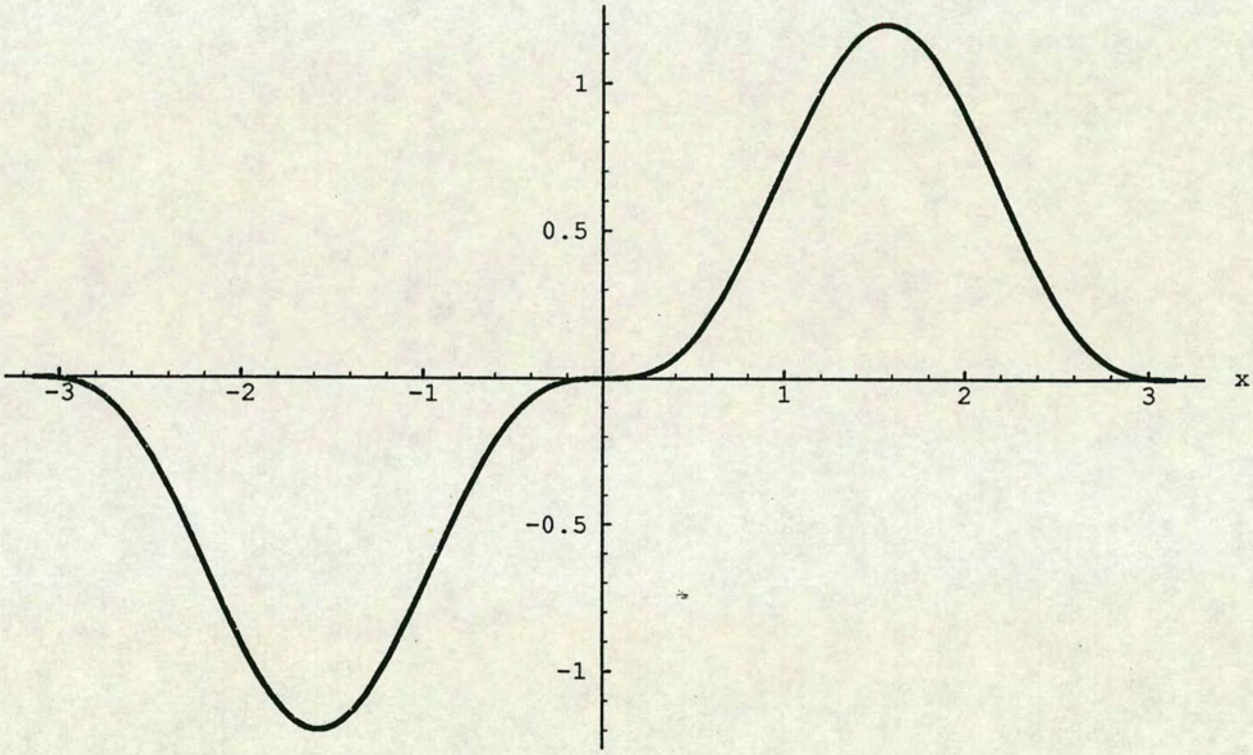

Figure 6.3: $f_y(x) = \frac{9}{10}\sin(x) - \frac{3}{10}\sin(3x)$, $\cos^2\frac{y}{2} = \frac{7}{10}$

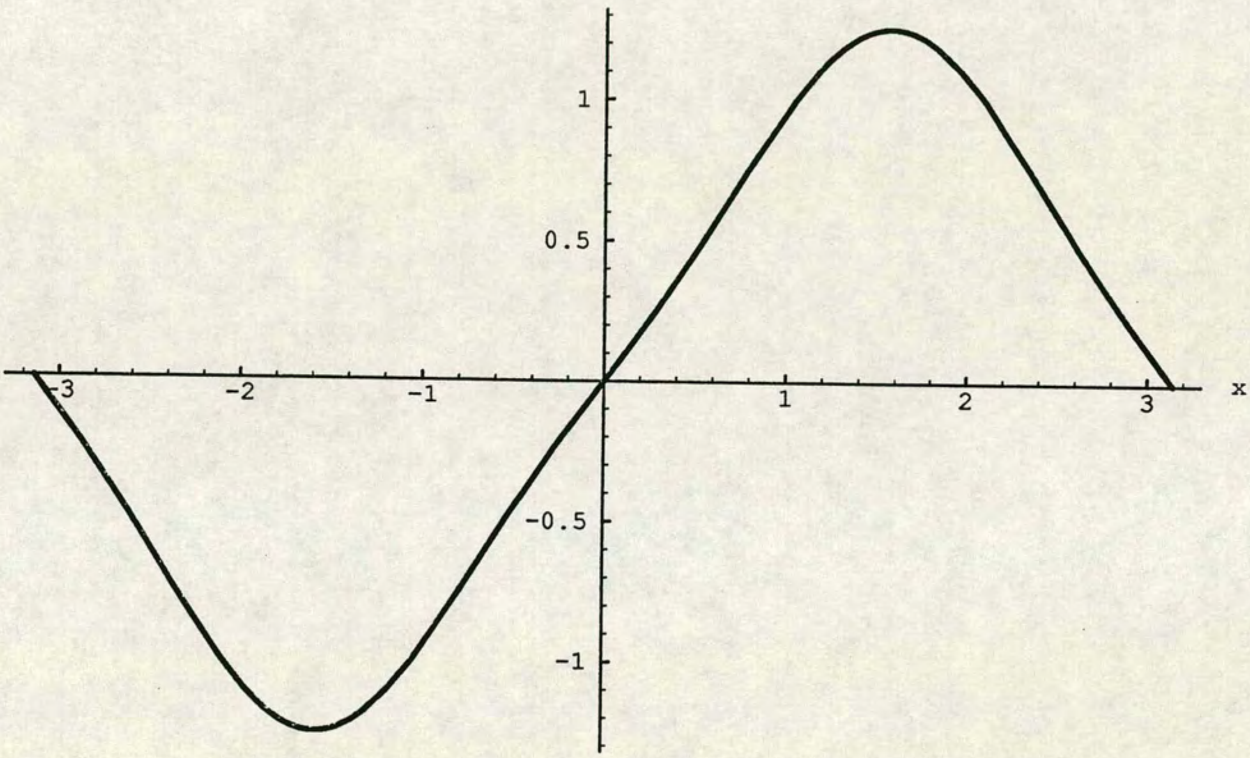

Figure 6.4: $f_y(x) = \frac{81}{70}\sin(x) - \frac{1}{10}\sin(3x)$, $\cos^2\frac{y}{2} = \frac{9}{10}$

Chapter 7

The Homotopy Type of the Space of Smooth Morse Functions on the Circle

Of interest is the homotopy type of the space of smooth real-valued functions on S^1 all of whose singularities are of codimension less than or equal to some positive integer k. Arnol'd has studied the case $k = 2$ (see Chapter 8). The cases $k \geq 3$ have been studied by Vasil'ev. We use his results in Chapter 9. In this chapter we study the case $k = 1$ (Morse functions).

7.1 Smooth Real-Valued Functions on S^1

Notation 7.1.1 *Denote $C^\infty(S^1, \mathbf{R})$ by Ω.*

Definition 7.1.2 *For every $k \geq 2$, let Σ_k be*

$$\left\{ g \in \Omega \,:\, g^{[1]}(\exp(i\theta)) = \ldots = g^{[k]}(\exp(i\theta)) = 0 \text{ for some } \exp(i\theta) \in S^1 \right\}.$$

(By $g^{[i]}$ we mean the i-th derivative.)

Definition 7.1.3 *Let Ω_k be the complement of Σ_k in Ω.*

We give Ω_k the subspace topology. Note that Ω_2 is the space of smooth Morse functions on S^1.

Definition 7.1.4 *For every positive integer l, define the subspace $\Omega_2(l)$ of Ω_2 by $f \in \Omega_2(l)$ iff $f \in \Omega_2$ and f has exactly l maxima.*

It follows immediately that

$$\Omega_2 = \coprod_{l \geq 1} \Omega_2(l).$$

It is proved in §7.3 that the path components of the space Ω_2 are the subspaces $\Omega_2(l)$, $l \geq 1$.

7.2 The Path Components of Ω_2

Lemma 7.2.1 *If the functions f and g in Ω_2 lie in the same path component then f and g have the same number of maxima.*

Proof If the functions f and g in Ω_2 lie in the same path component then there exists a continuous map

$$H \ : \ S^1 \times I \ \to \ \mathbf{R}$$

where I is the interval $[0, 1]$, such that

$$H(\exp(i\theta), 0) = f(\exp(i\theta)),$$

$$H(\exp(i\theta), 1) = g(\exp(i\theta))$$

and if

$$H_t(\exp(i\theta)) = H(\exp(i\theta), t)$$

then $H_t \in \Omega_2$ for every t in I.

Let $\tilde{H}$ be the restriction of H to $S^1 \times (0,1)$. Then $\tilde{H}$ is a parametrized Morse function. Recall the critical graph

$$\Gamma_{\tilde{H}} = \left\{ (\exp(i\theta), t) \in S^1 \times (0,1) \ : \ \tfrac{\partial \tilde{H}(\exp(i\theta),t)}{\partial \theta} = 0 \right\}.$$

By Proposition 2.1.3 and Corollary 2.1.4, each connected component of the covering space $\Gamma_{\tilde{H}}$ is diffeomorphic to $(0,1)$ and hence intersects each fibre in one point only. Hence the number of maxima of H_t is the same for every t in $(0,1)$. By continuity, this is the number of maxima of f and of g, since f and g are Morse. □

Corollary 7.2.2 *Each path component of Ω_2 is contained in $\Omega_2(l)$, for some $l \geq 1$.*

7.3 The Homotopy Type of $\Omega_2(l)$

We prove that for every positive integer l, the spaces $\Omega_2(l)$ and S^1 are homotopy equivalent by constructing an explicit homotopy equivalence. For purposes of calculation, we identify $\mathbf{R}/2\pi\mathbf{Z}$ with S^1 by the canonical map

$$\theta \mapsto \exp(i\theta)$$

and consider h in Ω as a smooth function, $h : \mathbf{R} \to \mathbf{R}$, periodic, with period 2π. For every positive integer l we find a continuous map

$$H \ : \ \Omega_2(l) \times I \ \to \ \Omega_2(l)$$

such that for every h in $\Omega_2(l)$

$$H(h,0) = h,$$

$$H(h,1) \in \{\theta \mapsto \cos(l\theta - \phi) \ : \ \phi \in \mathbf{R}/2\pi\mathbf{Z}\}$$

and for every ϕ in $\mathbf{R}/2\pi\mathbf{Z}$ and t in I

$$H(\theta \mapsto \cos(l\theta - \phi), t) = \theta \mapsto \cos(l\theta - \phi).$$

The homotopy H is constructed in four steps.

Step 1 For every a, b in $\mathbf{R}$, such that $a < b$, denote by ${\Omega_2}^{a,b}(l)$ the subspace

$$\{f \in \Omega_2(l) \; : \; f \text{ has value } a \text{ at all minima and value } b \text{ at all maxima}\}.$$

We construct a continuous map

$$H_1 \; : \; \Omega_2(l) \times I \; \to \; \Omega_2(l)$$

such that for every h in $\Omega_2(l)$

$$H_1(h, 0) = h,$$

$$H_1(h, 1) \in {\Omega_2}^{-1,+1}(l)$$

and for every ϕ in $\mathbf{R}/2\pi\mathbf{Z}$ and t in I

$$H_1(\theta \mapsto \cos(l\theta - \phi), t) = \theta \mapsto \cos(l\theta - \phi).$$

In order to construct the homotopy H_1 we define a smooth map

$$\begin{array}{rccl} p & : & \Omega_2(l) & \to & {\Omega_2}^{0,+1}(l) \\ & & h & \mapsto & p(h) \end{array}$$

such that $p(h)$ has a maximum at θ iff h has a maximum at θ and $p(h)$ has a minimum at θ iff h has a minimum at θ.

Calculating p(h) We define two continuous maps

$$E_U, \, E_L \; : \; \Omega_2(l) \; \to \; \Omega$$

such that for any h in $\Omega_2(l)$, the graph of the upper curve $E_U(h)$ lies above the graph of h and $E_U(h)(\theta) = h(\theta)$ iff h has a maximum at θ and the graph of the lower curve $E_L(h)$ lies below the graph of h and $E_L(h)(\theta) = h(\theta)$ iff h has a minimum at θ. The graphs of the curves $E_U(h)$ and $E_L(h)$ define an envelope for the graph of h. Later we shall set $p(h)(\theta)$ equal to

$$\frac{h(\theta) - E_L(h)(\theta)}{E_U(h)(\theta) - E_L(h)(\theta)}$$

(see Figure 7.3).

Lemma 7.3.1 *Let*

$$\lambda \ : \ \mathbf{R} \ \rightarrow \ \mathbf{R}$$

be defined by

$$\lambda(s) = 0$$

if $s \leq 0$ *and by*

$$\lambda(s) = \exp(-1/s)$$

if $s > 0$. *Then* $0 \leq \lambda(s) \leq 1$ *for every* s *in* $\mathbf{R}$ *and* λ *is smooth everywhere (see Figure 7.1).*

For strictly positive ϵ let $\Phi_\epsilon \ : \ \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$\Phi_\epsilon(s) = \frac{\lambda(s)}{\lambda(s) + \lambda(\epsilon - s)}.$$

Then Φ_ϵ is smooth, $0 \leq \Phi_\epsilon(s) \leq 1$ for every s in $\mathbf{R}$, $\Phi_\epsilon(s) = 0$ iff $s \leq 0$, $\Phi_\epsilon(s) = 1$ iff $s \geq \epsilon$ and $\frac{d\Phi_\epsilon(s)}{ds} > 0$ for s in $(0, \epsilon)$ (see Figure 7.2). For proof see [6, page 22].

Let $\{\Theta_i \ : \ \Theta_i \in \mathbf{R},\ i \in \mathbf{Z},\ i \text{ odd}\}$ be the critical points at which h has a maximum. Let $\{\Theta_i \ : \ \Theta_i \in \mathbf{R},\ i \in \mathbf{Z},\ i \text{ even}\}$ be the critical points at which h has a minimum, such that

$$\ldots < \Theta_{i-1} < \Theta_i < \Theta_{i+1} < \ldots$$

and $\Theta_{2l+i} = \Theta_i + 2\pi$, for every i in $\mathbf{Z}$. Let $h(\Theta_i) = \Psi_i$, for every i in $\mathbf{Z}$ (see Figure 7.3).

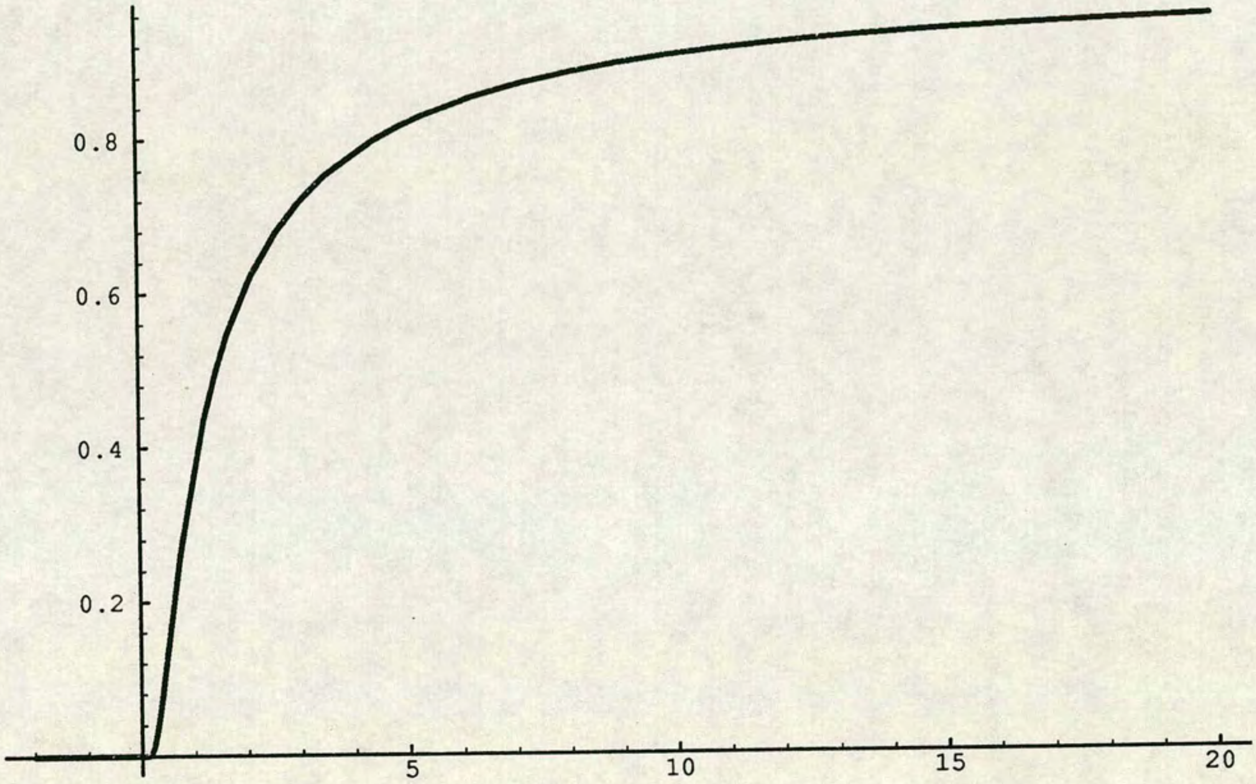

Figure 7.1: The Graph of $\lambda : [-1, 20] \to \mathbb{R}$

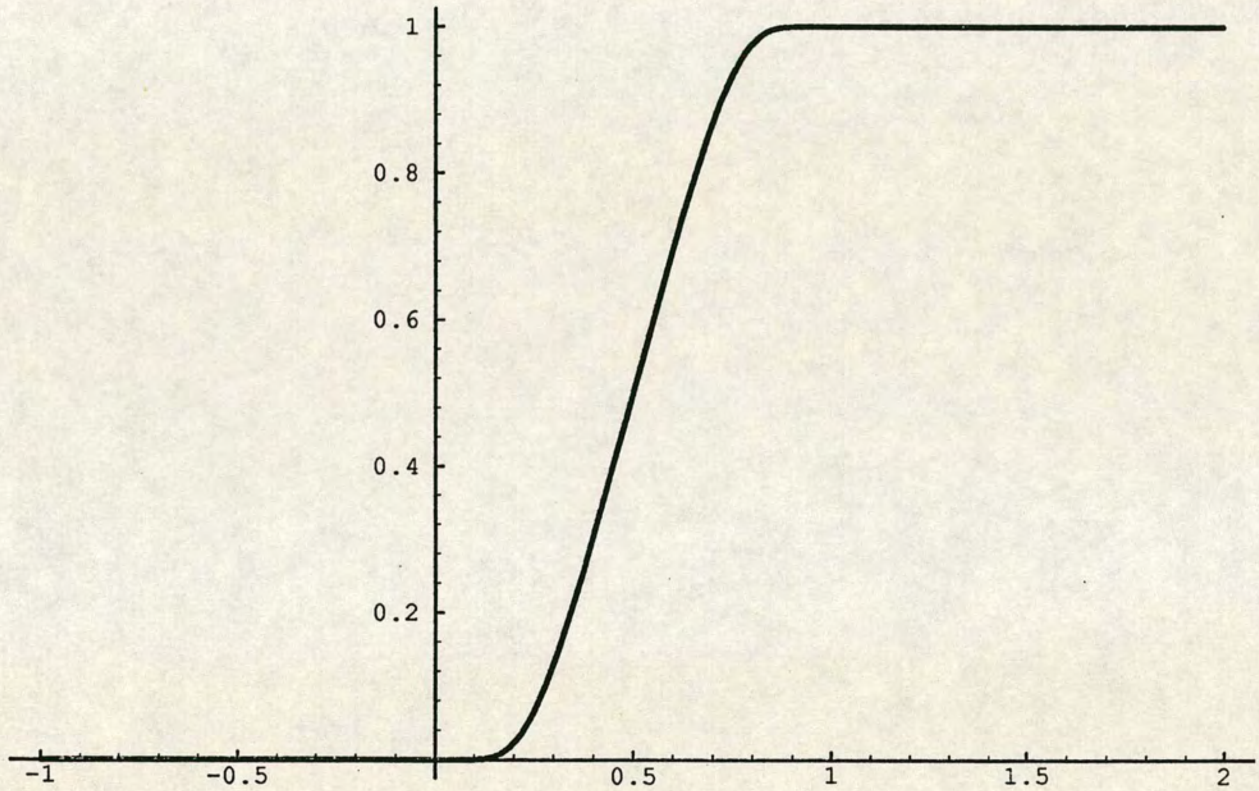

Figure 7.2: The Graph of $\Phi_1 : [-1, 2] \to \mathbb{R}$

Define ν_i by

$$\nu_i = 2\Psi_{i-1} - \Psi_{i+1} \quad \text{if } \Psi_{i-1} \geq \Psi_{i+1}$$
$$\nu_i = 2\Psi_{i+1} - \Psi_{i-1} \quad \text{if } \Psi_{i+1} \geq \Psi_{i-1}$$

for every even i. Then $\nu_i \geq max\{\Psi_{i-1}, \Psi_{i+1}\}$.

Define ν_i by

$$\nu_i = 2\Psi_{i-1} - \Psi_{i+1} \quad \text{if } \Psi_{i-1} \leq \Psi_{i+1}$$
$$\nu_i = 2\Psi_{i+1} - \Psi_{i-1} \quad \text{if } \Psi_{i+1} \leq \Psi_{i-1}$$

for every odd i. Then $\nu_i \leq min\{\Psi_{i-1}, \Psi_{i+1}\}$.

We join (Θ_i, Ψ_i) to $(\Theta_{i+1}, \nu_{i+1})$ and $(\Theta_{i+1}, \nu_{i+1})$ to $(\Theta_{i+2}, \Psi_{i+2})$, smoothly, using functions of the form of Φ_ϵ, this yields $E_U(h)$, i odd, and $E_L(h)$, i even.

For i odd and $\Psi_i \geq \Psi_{i+2}$, define

$$E_U(h)(\theta) = \Psi_i + (\Psi_i - \Psi_{i+2})\Phi_{(\Theta_{i+1}-\Theta_i)}(\theta - \Theta_i)$$

for θ in $[\Theta_i, \Theta_{i+1}]$ and

$$E_U(h)(\theta) = (2\Psi_i - \Psi_{i+2}) + (2\Psi_{i+2} - 2\Psi_i)\Phi_{(\Theta_{i+2}-\Theta_{i+1})}(\theta - \Theta_{i+1})$$

for θ in $[\Theta_{i+1}, \Theta_{i+2}]$.

For i odd and $\Psi_i \leq \Psi_{i+2}$, define

$$E_U(h)(\theta) = \Psi_i + (2\Psi_{i+2} - 2\Psi_i)\Phi_{(\Theta_{i+1}-\Theta_i)}(\theta - \Theta_i)$$

for θ in $[\Theta_i, \Theta_{i+1}]$ and

$$E_U(h)(\theta) = (2\Psi_{i+2} - \Psi_i) + (\Psi_i - \Psi_{i+2})\Phi_{(\Theta_{i+2}-\Theta_{i+1})}(\theta - \Theta_{i+1})$$

for θ in $[\Theta_{i+1}, \Theta_{i+2}]$.

For i even and $\Psi_{i+2} \geq \Psi_i$, define

$$E_L(h)(\theta) = \Psi_i + (\Psi_i - \Psi_{i+2})\Phi_{(\Theta_{i+1}-\Theta_i)}(\theta - \Theta_i)$$

for θ in $[\Theta_i, \Theta_{i+1}]$ and

$$E_L(h)(\theta) = (2\Psi_i - \Psi_{i+2}) + (2\Psi_{i+2} - 2\Psi_i)\Phi_{(\Theta_{i+2}-\Theta_{i+1})}(\theta - \Theta_{i+1})$$

for θ in $[\Theta_{i+1}, \Theta_{i+2}]$.

For i even and $\Psi_{i+2} \leq \Psi_i$, define

$$E_L(h)(\theta) = \Psi_i + (2\Psi_{i+2} - 2\Psi_i)\Phi_{(\Theta_{i+1}-\Theta_i)}(\theta - \Theta_i)$$

for θ in $[\Theta_i, \Theta_{i+1}]$ and

$$E_L(h)(\theta) = (2\Psi_{i+2} - \Psi_i) + (\Psi_i - \Psi_{i+2})\Phi_{(\Theta_{i+2}-\Theta_{i+1})}(\theta - \Theta_{i+1})$$

for θ in $[\Theta_{i+1}, \Theta_{i+2}]$.

Clearly, the maps $E_U(h)$ and $E_L(h)$ depend continuously on h.

Figure 7.3 is an example of a sketch of the graphs of $E_U(h)$ and $E_L(h)$ for a Morse function $h : S^1 \to \mathbf{R}$ with four critical points.

Let

$$p(h)(\theta) = \frac{h(\theta) - E_L(h)(\theta)}{E_U(h)(\theta) - E_L(h)(\theta)}.$$

The function $p(h)$ is smooth and periodic, with period 2π, since h, $E_U(h)$ and $E_L(h)$ are smooth and periodic, with period 2π. It follows from the definitions of $p(h)$, $E_U(h)$ and $E_L(h)$ that $p(h)(\Theta_i) = 0$ iff i is even, $p(h)(\Theta_i) = 1$ iff i is odd and that $0 < p(h)(\theta) < 1$ iff θ is a regular point of h. We prove that $p(h)$ is a Morse function with minima at Θ_i, i even, maxima at Θ_i, i odd, and no other critical points.

Assume $\Psi_{i'} \geq \Psi_{i'+2}$, for some fixed i' in $\mathbf{Z}$, i' odd (maxima), and that $\Psi_{i'+1} \geq \Psi_{i'-1}$, (minima). Explicitly, $p(h)(\theta)$ is equal to

$$\frac{h(\theta) - (2\Psi_{i'-1} - \Psi_{i'+1}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})\Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta - \Theta_{i'})}{\Psi_{i'} - (2\Psi_{i'-1} - \Psi_{i'+1}) + \Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta - \Theta_{i'})[(\Psi_{i'} - \Psi_{i'+2}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})]}$$

for θ in $[\Theta_{i'}, \Theta_{i'+1}]$.

The first derivative $\frac{dp(h)(\theta)}{d\theta}$ is equal to A/B where the numerator A is equal to

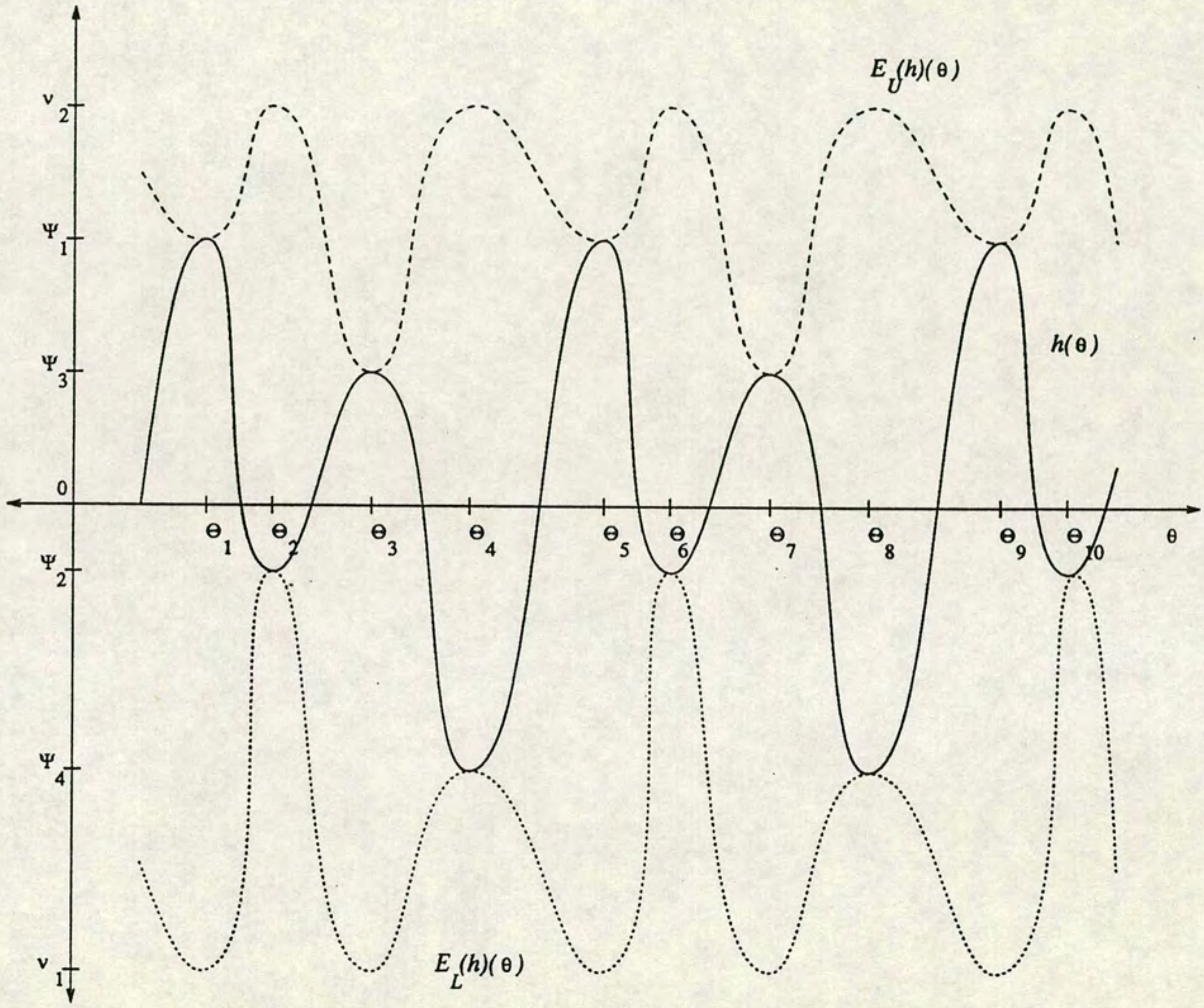

Figure 7.3: An Envelope for the Graph of h

$$\{\Psi_{i'} - (2\Psi_{i'-1} - \Psi_{i'+1}) + \Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta - \Theta_{i'})[(\Psi_{i'} - \Psi_{i'+2}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})]\}$$
$$\{\tfrac{dh}{d\theta} - (2\Psi_{i'+1} - 2\Psi_{i'-1})\tfrac{d\Phi}{d\theta}\}$$
$$-\{h(\theta) - (2\Psi_{i'-1} - \Psi_{i'+1}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})\Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta - \Theta_{i'})\}$$
$$\{[(\Psi_{i'} - \Psi_{i'+2}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})]\tfrac{d\Phi}{d\theta}\}$$

and the denominator B is equal to

$$\{\Psi_{i'} - (2\Psi_{i'-1} - \Psi_{i'+1}) + \Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta - \Theta_{i'})[(\Psi_{i'} - \Psi_{i'+2}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})]\}^2$$

for θ in $[\Theta_{i'}, \Theta_{i'+1}]$.

The denominator B is positive for θ in $[\Theta_{i'}, \Theta_{i'+1}]$. Rearranging, the numerator A is equal to

$$\{\Psi_{i'} - (2\Psi_{i'-1} - \Psi_{i'+1}) + \Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta - \Theta_{i'})[(\Psi_{i'} - \Psi_{i'+2}) - (2\Psi_{i'+1} - 2\Psi_{i'-1})]\}$$
$$\{\tfrac{dh}{d\theta}\} - \tfrac{d\Phi}{d\theta}(2\Psi_{i'+1} - 2\Psi_{i'-1})\{\Psi_{i'} - (2\Psi_{i'-1} - \Psi_{i'+1}) + (2\Psi_{i'-1} - \Psi_{i'+1}) - h(\theta)\}$$
$$-\tfrac{d\Phi}{d\theta}(\Psi_{i'} - \Psi_{i'+2})(h(\theta) - (2\Psi_{i'-1} - \Psi_{i'+1})).$$

The first term is

$$(E_U(h)(\theta) - E_L(h)(\theta))\,\tfrac{dh}{d\theta}$$

which has the sign of $\frac{dh}{d\theta}$ in $(\Theta_{i'}, \Theta_{i'+1})$, negative in our case. Recall $\frac{d\Phi_{(\Theta_{i'+1}-\Theta_{i'})}(\theta-\Theta_{i'})}{d\theta} > 0$ in $(\Theta_{i'}, \Theta_{i'+1})$, $\Psi_{i'} \geq \Psi_{i'+2}$ and $\Psi_{i'+1} \geq \Psi_{i'-1}$. Hence $\Psi_{i'} - \Psi_{i'+2} \geq 0$, $h(\theta) - (2\Psi_{i'-1} - \Psi_{i'+1}) > 0$, $(2\Psi_{i'+1} - 2\Psi_{i'-1}) \geq 0$ and $\Psi_{i'} - h(\theta) \geq 0$. Hence $\frac{dp(h)(\theta)}{d\theta} < 0$ for θ in $(\Theta_{i'}, \Theta_{i'+1})$.

Similarly, it may be shown that everywhere

$$\tfrac{dp(h)(\theta)}{d\theta} < 0 \text{ if } \tfrac{dh}{d\theta} < 0$$

and

$$\tfrac{dp(h)(\theta)}{d\theta} > 0 \text{ if } \tfrac{dh}{d\theta} > 0$$

and so

$$\tfrac{dp(h)(\theta)}{d\theta} = 0 \text{ iff } \tfrac{dh}{d\theta} = 0 \text{ iff } \theta = \Theta_i$$

for some i in $\mathbf{Z}$.

We show $\frac{d^2p(h)(\theta)}{d\theta^2}$ is non-zero at critical points of $p(h)$. Write

$$p(h) = f/g$$

where

$$f(\theta) = h(\theta) - E_L(h)(\theta)$$

and

$$g(\theta) = E_U(h)(\theta) - E_L(h)(\theta).$$

Then

$$\frac{d^2p(h)(\theta)}{d\theta^2} = \frac{gf' - g'f}{g^2}$$

and

$$\frac{d^2p(h)(\theta)}{d\theta^2} = \frac{g^2(f''g + g'f' - g'f' - g''f) - 2gg'(gf' - fg')}{g^4}$$

so at critical points Θ_i, i in $\mathbb{Z}$,

$$\frac{d^2p(h)(\theta)}{d\theta^2} = \frac{f''g - g''f}{g^2} = \frac{f''}{g} = \frac{\frac{d^2h}{d\theta^2}}{g}.$$

Now $g(\theta) = E_U(h)(\theta) - E_L(h)(\theta)$, always positive. So the sign of $\frac{d^2p(h)(\theta)}{d\theta^2}$ is the sign of $\frac{d^2h}{d\theta^2}$ at critical points. Hence $p(h)$ is a Morse function with a maximum of 1 at each Θ_i, i odd, and a minimum of 0 at each Θ_i, i even, and no other critical points.

Define H_1 by

$$H_1(h,t) = (1-t)h + 2tp(h) - t$$

for every h in $\Omega_2(l)$ and every t in I. If $h(\theta) = \cos(l\theta - \phi)$ then

$$p(h) = \frac{h+1}{2}$$

and so

$$H_1(h,t) = h$$

for every t in I. Clearly, H_1 is continuous.

Step 2 For every a, b in $\mathbf{R}$, such that $a < b$, denote by $\tilde{\Omega_2}^{a,b}(l)$ the subspace

$$\{f \in \Omega_2{}^{a,b}(l) \text{ such that } f \text{ has evenly spaced critical points}\}.$$

We construct a continuous map

$$H_2 \ : \ \Omega_2{}^{-1,+1}(l) \times I \ \rightarrow \ \Omega_2{}^{-1,+1}(l)$$

such that for every h in $\Omega_2{}^{-1,+1}(l)$

$$H_2(h, 0) = h,$$

$$H_2(h, 1) \in \tilde{\Omega_2}^{-1,+1}(l)$$

and for every ϕ in $\mathbf{R}/2\pi\mathbf{Z}$ and every t in I,

$$H_2(\theta \mapsto \cos(l\theta - \phi), t) = \theta \mapsto \cos(l\theta - \phi).$$

Choose h in $\Omega_2{}^{-1,+1}(l)$ and consider $\{\Theta_i \ : \ i \in \mathbf{Z}\}$, the set of critical points of h. Recall $\Theta_i < \Theta_{i+1}$ and $\Theta_i + 2\pi = \Theta_{i+2l}$ for every i in $\mathbf{Z}$. Consider the map

$$\Theta_i \mapsto \tilde{\Theta}_i$$

where

$$\tilde{\Theta}_{i-1+l} = \frac{\Sigma_{j=1}^{2l} \Theta_{i-1+j}}{2l} - \frac{\pi}{2l}$$

and

$$\tilde{\Theta}_{i+1} - \tilde{\Theta}_i = \frac{\pi}{l}$$

for every i in $\mathbf{Z}$.

The values $\tilde{\Theta}_i$, i in $\mathbf{Z}$ are equally spaced and form, in a sense, the average of the values Θ_i, i in $\mathbf{Z}$.

In order to define the homotopy H_2 we require a continuous map

$$\beta \ : \ \Omega_2{}^{-1,+1}(l) \rightarrow \mathrm{Diff}(S^1)$$

such that

$$\beta(h)(\Theta_i) = \tilde{\Theta}_i,$$

for every i in $\mathbf{Z}$.

It is convenient to identify $\mathrm{Diff}(S^1)$ with the subspace of $\mathrm{Diff}(\mathbf{R})$ such that for every element α in $\mathrm{Diff}(\mathbf{R})$ and every θ in $\mathbf{R}$

$$\alpha(\theta + 2\pi) = \alpha(\theta) + 2\pi.$$

Choose h in $\Omega_2{}^{-1,+1}(l)$. To construct the function

$$\beta(h) \ : \ \mathbf{R} \ \to \ \mathbf{R}$$

we take an average of two functions we call $\beta_1(h)$ and $\beta_2(h)$.

To construct the functions $\beta_1(h)$ and $\beta_2(h)$ we join the pairs $(\Theta_i, \tilde{\Theta}_i)$ and $(\Theta_{i+1}, \tilde{\Theta}_{i+1})$ by graphs of functions of the form of Φ_ϵ and its inverse (on $[0, \epsilon]$). If we were to simply link the pairs with the graph of

$$\theta \mapsto \tilde{\Theta}_i + (\tilde{\Theta}_{i+1} - \tilde{\Theta}_i)\Phi_{(\Theta_{i+1}-\Theta_i)}(\theta - \Theta_i),$$

$\Theta_i \leq \theta \leq \Theta_{i+1}$, for every i in $\mathbf{Z}$, the resulting function would not be invertible at $\{\Theta_i \ : \ i \in \mathbf{Z}\}$.

To avoid critical points we average two functions. Define

$$\begin{array}{rcll} \beta_1(h) \ : \ \mathbf{R} & \to & \mathbf{R} & \text{piecewise by} \\ \theta & \mapsto & \tilde{\Theta}_i + (\tilde{\Theta}_{i+1} - \tilde{\Theta}_i)\Phi_{(\Theta_{i+1}-\Theta_i)}(\theta - \Theta_i), & \end{array}$$

$\Theta_i \leq \theta \leq \Theta_{i+1}$, for every $i \in \mathbf{Z}$.

Define

$$\begin{array}{rcll} \beta_2(h) \ : \ \mathbf{R} & \to & \mathbf{R} & \text{piecewise by} \\ \theta & \mapsto & (\Phi_{\pi/l})^{-1}\left(\frac{\theta-\Theta_i}{\Theta_{i+1}-\Theta_i}\right) + \tilde{\Theta}_i, & \end{array}$$

$\Theta_i \leq \theta \leq \Theta_{i+1}$, for every $i \in \mathbf{Z}$. Here we take $[0, \pi/l]$ as the domain of $\Phi_{\pi/l}$ to ensure that it is one-to-one and onto. Note that $\beta_1(h)$ and $\beta_2(h)$ are smooth with positive first derivative at all regular points of h and are one-to-one and onto. At critical points of h, every derivative of $\beta_1(h)$ is zero and $\beta_2(h)$ is not differentiable. The graph of $\beta_j(h)$ intersects any diagonal line with gradient -1 in exactly one point, $j = 1, 2$.

Define $\beta(h) : \mathbf{R} \to \mathbf{R}$ by averaging the positions of the intercepts of the two curves $\beta_1(h)$ and $\beta_2(h)$ on the diagonal lines with gradient -1.

The graph of $\beta(h)$ tends smoothly to the diagonal with gradient $+1$ through every pair $(\Theta_i, \tilde{\Theta}_i)$ and is smooth elsewhere with positive first derivative. This can be visualized by rotating the graphs of $\beta_1(h)$ and $\beta_2(h)$ by $\pi/4$ clockwise, taking the average and rotating back by $\pi/4$ anticlockwise. As $\beta(h)$ is defined piecewise it is clear that

$$\beta(h)(\theta + 2\pi) = \beta(h)(\theta) + 2\pi$$

for every θ in $\mathbf{R}$. Hence $\beta(h)$ induces a diffeomorphism of S^1 satisfying the restriction $\beta(h)(\Theta_i) = \tilde{\Theta}_i$, for every i in $\mathbf{Z}$, as required.

We now construct the homotopy H_2. Let

$$\begin{array}{rcl} \beta(h,t) \ : \ \Omega_2{}^{-1,+1}(l) \times I & \to & \mathrm{Diff}(\mathbf{R}) \\ (h,t) & \mapsto & t\beta(h) + (1-t)(\mathrm{Identity}). \end{array} \qquad \text{be defined by}$$

Let

$$H_2(h,t) = \left\{\theta \mapsto h \circ \left(\beta(h,t)^{-1}(\theta)\right)\right\}$$

for every h in $\Omega_2{}^{-1,+1}(l)$ and every t in I. Now,

$$\frac{dH_2(h,t)}{d\theta} = \frac{dh}{d\theta}\frac{d\left(\beta(h,t)^{-1}(\theta)\right)}{d\theta},$$

a positive multiple of $\frac{dh}{d\theta}$. When $\frac{dh}{d\theta}$ is zero,

$$\frac{d^2H_2(h,t)}{d\theta^2} = \frac{d^2h}{d\theta^2}\frac{d\left(\beta(h,t)^{-1}(\theta)\right)}{d\theta},$$

a positive multiple of $\frac{d^2h}{d\theta^2}$.

Hence $H_2(h,t)$ is a Morse function, periodic, with period 2π, with a maximum of 1 at θ iff h has a maximum at $(\beta(h,t)^{-1}(\theta))$ and a minimum of -1 at θ iff h has a minimum at $(\beta(h,t)^{-1}(\theta))$. Hence $H_2(h,t) \in {\Omega_2}^{-1,+1}(l)$ for every h in ${\Omega_2}^{-1,+1}(l)$ and t in I and

$$H_2(h,1) \in \tilde{\Omega_2}^{-1,+1}(l)$$

for every h in ${\Omega_2}^{-1,+1}(l)$. Moreover, the map

$$\theta \mapsto \cos(l\theta - \phi)$$

has evenly spaced critical points. Hence,

$$H_2(\theta \mapsto \cos(l\theta - \phi), t) = \theta \mapsto \cos(l\theta - \phi)$$

for every ϕ in $\mathbf{R}/2\pi\mathbf{Z}$ and every t in I. Clearly, H_2 is continuous.

Step 3 We construct a continuous map

$$H_3 \;:\; \tilde{\Omega_2}^{-1,+1}(l) \times I \;\rightarrow\; \tilde{\Omega_2}^{-1,+1}(l)$$

such that for every h in $\tilde{\Omega_2}^{-1,+1}(l)$

$$H_3(h,0) = h,$$

$$H_3(h,1) \in \{\theta \mapsto \cos(l\theta - \phi) \;:\; \phi \in \mathbf{R}/2\pi\mathbf{Z}\}$$

and for every ϕ in $\mathbf{R}/2\pi\mathbf{Z}$ and t in I

$$H_3(\theta \mapsto \cos(l\theta - \phi), t) = \theta \mapsto \cos(l\theta - \phi).$$

If h is in $\tilde{\Omega_2}^{-1,+1}(l)$ then the map

$$\begin{array}{rcrcl} \hat{h} & : & \mathbf{R} & \rightarrow & \mathbf{R} \\ & & \theta & \mapsto & h(\theta/l) \end{array} \qquad \text{defined by}$$

has period $2\pi l$ and critical points spaced π apart. Let $\Theta(h)$ be any critical point at which $\hat{h}$ has a maximum, it is defined $\mathrm{mod}(2\pi)$ and varies continuously with h. The map

$$\begin{array}{rcrcl} C & : & \tilde{\Omega_2}^{-1,+1}(l) & \rightarrow & \mathbf{R}/2\pi\mathbf{Z} \\ & & h & \mapsto & \Theta(h) \end{array} \quad \text{defined by}$$

is continuous and hence the map

$$\begin{array}{rcl} \tilde{C} \ : \ \tilde{\Omega_2}^{-1,+1}(l) & \to & \{\theta \mapsto \cos(l\theta - \phi) \ : \ \phi \in \mathbf{R}/2\pi\mathbf{Z}\} \\ h & \mapsto & \{\theta \mapsto \cos(l\theta - \Theta(h))\} \end{array}$$

is continuous. The functions h and $\tilde{C}(h)$ share the same critical points. They attain a maximum of $+1$ at $\{(\Theta(h) + 2\pi k)/l \ : \ k \in \mathbf{Z}\}$ and a minimum of -1 at $\{(\Theta(h) + \pi + 2\pi k)/l \ : \ k \in \mathbf{Z}\}$. Hence

$$H_3(h,t) = (1-t)h + t\tilde{C}(h)$$

is a Morse function for every h in $\tilde{\Omega_2}^{-1,+1}(l)$, and every t in I.

Step 4 We define a homotopy

$$H \ : \ \Omega_2(l) \times I \ \to \ \Omega_2(l) \ \ by$$

$$\begin{array}{ll} H(h,s) = H_1(h,3s) & \text{for } 0 \leq s \leq 1/3, \\ H(h,s) = H_2(H_1(h,1), 3(s-1/3)) & \text{for } 1/3 \leq s \leq 2/3 \text{ and} \\ H(h,s) = H_3(H_2(H_1(h,1),1), 3(s-2/3)) & \text{for } 2/3 \leq s \leq 1. \end{array}$$

The map H is continuous as H_1, H_2 and H_3 are continuous. The image of $\Omega_2(l) \times \{1\}$ is

$$\{\theta \mapsto \cos(l\theta - \phi) \ : \ \phi \in \mathbf{R}/2\pi\mathbf{Z}\},$$

diffeomorphic to S^1.

Theorem 7.3.2 *For every positive integer l, the spaces $\Omega_2(l)$ and S^1 are homotopy equivalent.*

Proof For every positive integer l let

$$\begin{array}{rcll} f \ : \ \Omega_2(l) & \to & S^1 & \text{be defined by} \\ h & \mapsto & \exp(i\Theta(H(h,1))) & \text{and} \end{array}$$

$$\begin{array}{rcll} g \ : \ S^1 & \to & \Omega_2(l) & \text{be defined by} \\ \exp(i\phi) & \mapsto & (\theta \mapsto (\cos(l\theta - \phi))). & \end{array}$$

Then

$$g \circ f = H(.,1) \ : \ \Omega_2(l) \to \Omega_2(l)$$

is homotopic to the identity map and

$$f \circ g : S^1 \to S^1$$

is the identity map. Hence the spaces $\Omega_2(l)$ and S^1 are homotopy equivalent. □

Corollary 7.3.3 *For every positive integer l, the space $\Omega_2(l)$ is path connected.*

Corollary 7.3.4 *The path components of the space Ω_2 are the subspaces $\Omega_2(l)$, $l \geq 1$.*

Corollary 7.3.5 *The space Ω_2 is homotopy equivalent to $\mathbb{N} \times S^1$.*

Chapter 8

The Fundamental Group of the Space of Generalized Morse Functions on the Circle

Generalized Morse functions were defined in §4.6. Recall, f in Ω, the space of smooth real-valued functions on the circle, is a generalized Morse function if every singularity of f is either of type A_1 or of type A_2. Following the notation of the previous chapter, denote by Ω_3 the space of smooth generalized Morse functions on the circle. We prove that Ω_3 is path connected and that the fundamental group of Ω_3 is isomorphic to $\mathbf{Z}$. The main ideas in this chapter have been taken from Arnol'd's paper [4]. The details have been reconstructed.

8.1 The Path Components of Ω_3

Lemma 8.1.1 *The space Ω_3 is path connected.*

Proof Recall from §4.5 that a function in $C^\infty(\mathbf{R},\mathbf{R})$ with an A_2 singularity may be perturbed locally either to eliminate the singularity or to replace it by a pair of nondegenerate critical points, namely one maximum and one minimum (and vice versa).

The same is true for elements of Ω provided that the perturbed function retains at least one pair of critical points at which it attains a maximum and a minimum. Elements of Ω have finite critical point sets since A_1 and A_2 singularities are isolated singularities and S^1 is compact.

We construct a continuous path in Ω_3 from a given element in Ω_3 to an element in $\Omega_2(1)$ (see Definition 7.1.4). Perturb the given function locally around each A_2 singularity to eliminate the critical point. Group the remaining nondegenerate critical points into pairs of adjacent critical points. Replace all pairs except one by A_2 singularities. Eliminate all the A_2 singularities, leaving just one pair of nondegenerate critical points.

By Corollary 7.3.4, the subspace $\Omega_2(1)$ is path connected. Hence Ω_3 is path connected. □

8.2 Closed Curves in the Torus

In §8.5 and §8.6 we show that both $\mathbf{Z}$ and $\pi_1(\Omega_3)$ are isomorphic to the equivalence classes of embedded closed curves in the torus with no horizontal tangential inflections. These curves and their equivalence classes will be defined precisely in §8.4. By horizontal we mean at constant ϕ (see next paragraph).

As in §7.3, identify $\mathbf{R}/2\pi\mathbf{Z}$ with S^1 by the canonical map

$$\omega \mapsto \exp(i\omega)$$

and consider any function in Ω as a smooth function on $\mathbf{R}$, periodic, with period 2π. Any map f in $C^\infty(S^1 \times S^1, \mathbf{R})$ determines a family in Ω, namely

$$\{f_\phi \in \Omega \, : \, \phi \in \mathbf{R}/2\pi\mathbf{Z}\},$$

where $f_\phi(\omega) = f(\omega, \phi)$, for every ω and ϕ in $\mathbf{R}/2\pi\mathbf{Z}$.

The property that f in $C^\infty(S^1 \times S^1, \mathbf{R})$ has only isolated critical points on fibres is generic (see Chapter 4). If f is such a function, the critical graph

$$\Gamma_f = \left\{(\omega, \phi) \,:\, \tfrac{df_\phi(\omega)}{d\omega} = 0\right\}$$

is a curve in $S^1 \times S^1$. The curve Γ_f has a singularity at (ω', ϕ') iff here $\frac{\partial^2 f}{\partial \omega^2} = \frac{\partial^2 f}{\partial \phi \partial \omega} = 0$. Elsewhere Γ_f is smooth. A singularity of Γ_f at (ω', ϕ') is called a double point singularity or a Morse type singularity (see [4]) if the Hessian of $\frac{\partial f}{\partial \omega}$ is nonsingular with respect to the variables ω and ϕ at (ω', ϕ'), that is, if the germ of $\frac{\partial f}{\partial \omega}$ at (ω', ϕ') is diffeomorphic to the germ $x^2 \pm y^2$ at the origin of the plane.

Lemma 8.2.1 *There exists a dense open subset R of $C^\infty(S^1 \times S^1, \mathbf{R})$ such that for every f in R, Γ_f is smooth everywhere and if $(\omega, \phi) \in \Gamma_f$ then either*

1. *f_ϕ has an A_1 singularity at ω or*
2. *f_ϕ has an A_2 singularity at ω.*

Proof Define the subset W_R of $J^3(S^1 \times S^1, \mathbf{R})$ by

$$W_R = \left\{j^3 f_{(\omega', \phi')} \,:\, f \in C^\infty(S^1 \times S^1, \mathbf{R}),\, \tfrac{\partial f}{\partial \omega} = 0,\, \tfrac{\partial^2 f}{\partial \omega^2} = 0 \text{ at } (\omega', \phi')\right\}.$$

Define the map

$$\begin{array}{rcrcll} \tilde{\pi} & : & J^3(S^1 \times S^1, \mathbf{R}) & \to & \mathbf{R}^2 & \qquad\qquad \text{by} \\ & & j^3 f_{(\omega', \phi')} & \mapsto & \left(\frac{\partial f}{\partial \omega}, \frac{\partial^2 f}{\partial \omega^2}\right)_{(\omega', \phi')}. & \end{array}$$

The map $\tilde{\pi}$ is a submersion, hence $W_R = \tilde{\pi}^{-1}(\underline{0})$ is a submanifold of $J^3(S^1 \times S^1, \mathbf{R})$ of codimension two.

By the Thom Transversality Theorem (Theorem 2.2.8)

$$R = \left\{f \in C^\infty(S^1 \times S^1, \mathbf{R}) \,:\, j^3 f \pitchfork W_R\right\}$$

is a residual subset of $C^\infty(S^1 \times S^1, \mathbf{R})$ in the Whitney C^∞ topology. Since $\{\underline{0}\}$ is closed in $\mathbf{R}^2$ and $\tilde{\pi}$ is continuous, W_R is closed in $J^3(S^1 \times S^1, \mathbf{R})$. By Proposition 2.2.15, R is open. By Proposition 2.2.4, R is dense. Hence R is dense and open.

Assume $j^3 f \pitchfork W_R$ at (ω', ϕ'). If $j^3 f_{(\omega',\phi')} \notin W_R$, then $f_{\phi'}$ is either regular or has an A_1 singularity at ω'. In the latter case, $\frac{\partial^2 f}{\partial \omega^2} \neq 0$ at (ω', ϕ') and so here Γ_f is smooth. If $j^3 f_{(\omega',\phi')} \in W_R$ then

$$\begin{array}{rcccl} \pi \circ j^3 f & : & S^1 \times S^1 & \rightarrow & \mathbf{R}^2 \qquad\qquad \text{defined by} \\ & & (\omega, \phi) & \mapsto & \left(\frac{\partial f}{\partial \omega}, \frac{\partial^2 f}{\partial \omega^2}\right) \end{array}$$

is a submersion at (ω', ϕ'). Hence the matrix

$$\left(\begin{array}{cc} \frac{\partial^2 f}{\partial \omega^2} & \frac{\partial^2 f}{\partial \phi \partial \omega} \\ \frac{\partial^3 f}{\partial \omega^3} & \frac{\partial^3 f}{\partial \phi \partial \omega^2} \end{array} \right)_{(\omega', \phi')}$$

is nonsingular which implies $\frac{\partial^2 f}{\partial \phi \partial \omega} \neq 0$ and $\frac{\partial^3 f}{\partial \omega^3} \neq 0$ at (ω', ϕ'). Hence Γ_f is smooth at (ω', ϕ') and $f_{\phi'}$ has an A_2 singularity at ω'. □

8.3 From Homotopies to Cobordisms

Consider a smooth homotopy

$$F : S^1 \times S^1 \times I \rightarrow \mathbf{R}.$$

Set $F^s_\phi(\omega) = F^s(\omega, \phi) = F(\omega, \phi, s)$ for every (ω, ϕ, s) in $S^1 \times S^1 \times I$. Then F^s_ϕ is in Ω, for every (ϕ, s) in $S^1 \times I$.

Let $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ be the set of all functions

$$f \ : \ S^1 \times S^1 \times I \ \rightarrow \ \mathbf{R}$$

which extend to smooth functions on $S^1 \times S^1 \times \tilde{I}$ where $\tilde{I}$ is a fixed open interval containing I.

Let I_1 be another fixed open interval containing I. We show that if

$$f \; : \; S^1 \times S^1 \times I \;\to\; \mathbf{R}$$

extends to a smooth function on $S^1 \times S^1 \times \tilde{I}$ then it extends to a smooth function on $S^1 \times S^1 \times I_1$. Let $I_2 = I_1 \cap \tilde{I}$. Then I_2 is an open interval containing I. Let $\tilde{f}$ be an extension of f to $S^1 \times S^1 \times \tilde{I}$. Let f_2 be the restriction of $\tilde{f}$ to $S^1 \times S^1 \times I_2$. Then

$$f_2 \; : \; S^1 \times S^1 \times I_2 \;\to\; \mathbf{R}$$

is smooth. Using functions of the form of Φ_ϵ (see §7.3), we can construct a smooth function

$$f_1 \; : \; S^1 \times S^1 \times I_1 \;\to\; \mathbf{R}$$

such that f_1 and f_2 agree on $S^1 \times S^1 \times I_2$. Hence f_1 and f agree on $S^1 \times S^1 \times I$. Hence f extends to a smooth function on $S^1 \times S^1 \times I_1$. Hence, the set $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ is independent of the choice of $\tilde{I}$. Let the map

$$p \; : \; C^\infty(S^1 \times S^1 \times \tilde{I}, \mathbf{R}) \;\to\; C^\infty(S^1 \times S^1 \times I, \mathbf{R})$$

be induced by restriction. Give $C^\infty(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ the Whitney C^∞ topology and $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ the quotient topology. Then p is both continuous and open and hence maps dense open subsets to dense open subsets.

Lemma 8.3.1 *There exists a dense open subset Q of $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ such that for every F in Q, the set of points (ω, ϕ, s) in $S^1 \times S^1 \times I$ at which Γ_{F^s} is not smooth is finite and at each of these points, Γ_{F^s} has a double point singularity .*

Proof Define the subset $W_{\tilde{Q}}$ of $J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ to be equal to

$$\left\{ j^3F_{(\omega', \phi', s')} \; : \; \tfrac{\partial F}{\partial \omega} = 0, \; \tfrac{\partial^2 F}{\partial \omega^2} = 0, \; \tfrac{\partial^2 F}{\partial \phi \partial \omega} = 0 \text{ at } (\omega', \phi', s') \right\}.$$

Define the map

$$\begin{array}{rcll} \tilde{\pi} & : & J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R}) & \rightarrow & \mathbf{R}^3 \\ & & j^3F_{(\omega',\phi',s')} & \mapsto & \left(\frac{\partial F}{\partial \omega}, \frac{\partial^2 F}{\partial \omega^2}, \frac{\partial^2 F}{\partial \omega \partial \phi}\right)_{(\omega',\phi',s')}. \end{array} \quad \text{by}$$

The map $\tilde{\pi}$ is a submersion everywhere, hence $W_{\tilde{Q}} = \tilde{\pi}^{-1}(\underline{0})$ is a submanifold of $J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ of codimension three.

By the Thom Transversality Theorem (Theorem 2.2.8)

$$\tilde{Q} = \left\{ F \in C^\infty(S^1 \times S^1 \times \tilde{I}, \mathbf{R}) \,:\, j^3F \pitchfork W_{\tilde{Q}} \right\}$$

is a residual subset of $C^\infty(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ in the Whitney C^∞ topology. Since $\{\underline{0}\}$ is closed in $\mathbf{R}^3$ and $\tilde{\pi}$ is continuous, $W_{\tilde{Q}}$ is closed in $J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$. By Proposition 2.2.15, $\tilde{Q}$ is open. By Proposition 2.2.4, $\tilde{Q}$ is dense. Hence $\tilde{Q}$ is dense and open.

For F in $\tilde{Q}$,

$$\left\{ (\omega', \phi', s') \in S^1 \times S^1 \times I \,:\, j^3F_{(\omega',\phi',s')} \in W_{\tilde{Q}} \right\}$$

is a finite set of points as $S^1 \times S^1 \times I$ is compact and of dimension three. Hence

$$\left\{ (\omega', \phi', s') \in S^1 \times S^1 \times I \,:\, \Gamma_{F^{s'}} \text{has a singularity at } (\omega', \phi') \right\}$$

is a finite set of points.

Assume that $j^3F \pitchfork W_{\tilde{Q}}$ at (ω', ϕ', s'). If $j^3F_{(\omega',\phi',s')} \notin W_{\tilde{Q}}$ then either $(\omega', \phi') \notin \Gamma_{F^{s'}}$ or $\Gamma_{F^{s'}}$ is smooth at (ω', ϕ'). If $j^3F_{(\omega',\phi',s')} \in W_{\tilde{Q}}$ then

$$\begin{array}{rcll} \pi \circ j^3F & : & S^1 \times S^1 \times \tilde{I} & \rightarrow & \mathbf{R}^3 \\ & & (\omega, \phi, s) & \mapsto & \left(\frac{\partial F}{\partial \omega}, \frac{\partial^2 F}{\partial \omega^2}, \frac{\partial^2 F}{\partial \omega \partial \phi}\right) \end{array} \quad \text{defined by}$$

is a submersion at (ω', ϕ', s'). Hence the matrix

$$\begin{pmatrix} \frac{\partial^2 F}{\partial \omega^2} & \frac{\partial^2 F}{\partial \phi \partial \omega} & \frac{\partial^2 F}{\partial \omega \partial s} \\ \frac{\partial^3 F}{\partial \omega^3} & \frac{\partial^3 F}{\partial \phi \partial \omega^2} & \frac{\partial^3 F}{\partial \omega^2 \partial s} \\ \frac{\partial^3 F}{\partial \omega^2 \partial \phi} & \frac{\partial^3 F}{\partial \omega \partial \phi^2} & \frac{\partial^3 F}{\partial \omega \partial \phi \partial s} \end{pmatrix}_{(\omega',\phi',s')}$$

is nonsingular which implies

$$\frac{\partial^2 F}{\partial\omega\partial s}\left\{\frac{\partial^3 F}{\partial\omega^3}\frac{\partial^3 F}{\partial\omega\partial\phi^2}-\left(\frac{\partial^3 F}{\partial\omega^2\partial\phi}\right)^2\right\}_{(\omega',\phi',s')}\neq 0.$$

Hence $\frac{\partial^3 F}{\partial\omega^3}\frac{\partial^3 F}{\partial\omega\partial\phi^2}-(\frac{\partial^3 F}{\partial\omega^2\partial\phi})^2\neq 0$ at (ω',ϕ',s') and so the Hessian of $\frac{\partial F^s}{\partial\omega}$ with respect to the variables ω and ϕ is nonsingular at (ω',ϕ'). Hence $\Gamma_{F^{s'}}$ has a double point singularity at (ω',ϕ').

The set $Q=p(\tilde{Q})$ is a dense open subset of $C^\infty(S^1\times S^1\times I,\mathbf{R})$ satisfying the requirements of the lemma. □

Lemma 8.3.2 *There exists a dense open subset M of $C^\infty(S^1\times S^1\times I,\mathbf{R})$ such that, for every F in M and every (ω,ϕ,s) in $S^1\times S^1\times I$, if $(\omega,\phi)\in\Gamma_{F^s}$ then either Γ_{F^s} is smooth at (ω,ϕ) or F^s_ϕ has an A_2 singularity at ω.*

Proof Define the subset $W_{\tilde{M}}$ of $J^4(S^1\times S^1\times\tilde{I},\mathbf{R})$ to be equal to

$$\left\{j^4F_{(\omega',\phi',s')}\,:\,\frac{\partial F}{\partial\omega}=0,\ \frac{\partial^2 F}{\partial\omega^2}=0,\ \frac{\partial^3 F}{\partial\omega^3}=0,\ \frac{\partial^2 F}{\partial\phi\partial\omega}=0 \text{ at } (\omega',\phi',s')\right\}.$$

Define the map

$$\begin{array}{rcll}\tilde{\pi} & : & J^4(S^1\times S^1\times\tilde{I},\mathbf{R}) \;\rightarrow\; \mathbf{R}^4 & \text{by}\\ & & j^4F_{(\omega',\phi',s')} \;\mapsto\; \left(\frac{\partial F}{\partial\omega},\frac{\partial^2 F}{\partial\omega^2},\frac{\partial^3 F}{\partial\omega^3},\frac{\partial^2 F}{\partial\omega\partial\phi}\right)_{(\omega',\phi',s')}. & \end{array}$$

The map $\tilde{\pi}$ is a submersion everywhere, hence $W_{\tilde{M}}=\tilde{\pi}^{-1}(\underline{0})$ is a submanifold of $J^4(S^1\times S^1\times\tilde{I},\mathbf{R})$ of codimension four.

By the Thom Transversality Theorem (Theorem 2.2.8)

$$\tilde{M}=\left\{F\in C^\infty(S^1\times S^1\times\tilde{I},\mathbf{R})\,:\,j^4F\pitchfork W_{\tilde{M}}\right\}$$

is a residual subset of $C^\infty(S^1\times S^1\times\tilde{I},\mathbf{R})$ in the Whitney C^∞ topology. Since $\{\underline{0}\}$ is closed in $\mathbf{R}^4$ and $\tilde{\pi}$ is continuous, $W_{\tilde{M}}$ is closed in $J^4(S^1\times S^1\times\tilde{I},\mathbf{R})$. By

Proposition 2.2.15, $\tilde{M}$ is open. By Proposition 2.2.4, $\tilde{M}$ is dense. Hence $\tilde{M}$ is dense and open.

For F in $\tilde{M}$, $j^4F(S^1 \times S^1 \times \tilde{I}) \cap W_{\tilde{M}} = \emptyset$. Hence if $F \in \tilde{M}$ and $(\omega, \phi) \in \Gamma_{F^s}$ then either Γ_{F^s} is smooth at (ω, ϕ) or F^s_ϕ has an A_2 singularity at ω.

The set $M = p(\tilde{M})$ is a dense open subset of $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ satisfying the requirements of the lemma. □

Lemma 8.3.3 *There exists a dense open subset P of $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ such that for every F in P, the set of points in $S^1 \times S^1 \times I$ at which Γ_{F^s} is not smooth is finite and elsewhere, if $(\omega, \phi) \in \Gamma_{F^s}$, then either*

1. *F^s_ϕ has an A_1 singularity at ω or*

2. *F^s_ϕ has an A_2 singularity at ω.*

Proof Define the subset $W_{\tilde{P}}$ of $J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ to be equal to

$$\left\{ j^3F_{(\omega',\phi',s')} \,:\, \tfrac{\partial F}{\partial \omega} = 0,\ \tfrac{\partial^2 F}{\partial \omega^2} = 0,\ \tfrac{\partial^2 F}{\partial \omega^2}\tfrac{\partial^3 F}{\partial \phi \partial \omega^2} - \tfrac{\partial^2 F}{\partial \phi \partial \omega}\tfrac{\partial^3 F}{\partial \omega^3} = 0 \text{ at } (\omega', \phi', s') \right\}.$$

Define the map

$$\begin{array}{rcll} \tilde{\pi} & : & J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R}) \;\to\; \mathbf{R}^3 & \text{by} \\ & & j^3F_{(\omega',\phi',s')} \;\mapsto\; \left(\tfrac{\partial F}{\partial \omega}, \tfrac{\partial^2 F}{\partial \omega^2}, \tfrac{\partial^2 F}{\partial \omega^2}\tfrac{\partial^3 F}{\partial \phi \partial \omega^2} - \tfrac{\partial^2 F}{\partial \phi \partial \omega}\tfrac{\partial^3 F}{\partial \omega^3}\right)_{(\omega',\phi',s')}. & \end{array}$$

The map $\tilde{\pi}$ is a submersion everywhere, hence $W_{\tilde{P}} = \tilde{\pi}^{-1}(\underline{0})$ is a submanifold of $J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ of codimension three.

By the Thom Transversality Theorem (Theorem 2.2.8)

$$\tilde{P} = \left\{ F \in C^\infty(S^1 \times S^1 \times \tilde{I}, \mathbf{R}) \,:\, j^3F \pitchfork W_{\tilde{P}} \right\}$$

is a residual subset of $C^\infty(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$ in the Whitney C^∞ topology. Since $\{\underline{0}\}$ is closed in $\mathbf{R}^3$ and $\tilde{\pi}$ is continuous, $W_{\tilde{P}}$ is closed in $J^3(S^1 \times S^1 \times \tilde{I}, \mathbf{R})$. By

Proposition 2.2.15, $\tilde{P}$ is open. By Proposition 2.2.4, $\tilde{P}$ is dense. Hence $\tilde{P}$ is dense and open.

For F in $\tilde{P}$,

$$\tilde{\tilde{P}} = \left\{(\omega', \phi', s') \in S^1 \times S^1 \times I \ : \ j^3F_{(\omega', \phi', s')} \in W_{\tilde{P}}\right\}$$

is a finite set of points as $S^1 \times S^1 \times I$ is compact. If Γ_{F^s}, s in I, has a singularity at (ω, ϕ) then $(\omega, \phi, s) \in \tilde{\tilde{P}}$. If $s \in I$ and if $(\omega, \phi, s) \notin \tilde{\tilde{P}}$, then $j^3F^s \pitchfork W_R$ at (ω, ϕ). By Lemma 8.2.1, $P = p(\tilde{P})$ is a dense open subset of $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ satisfying the requirements of the lemma. □

Corollary 8.3.4 *There exists a dense open subset of $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ such that for every element F, the set of points (ω, ϕ, s) in $S^1 \times S^1 \times I$ at which Γ_{F^s} is not smooth is finite and at each of these points, Γ_{F^s} has a double point singularity and F^s_ϕ has an A_2 singularity . Elsewhere, if $(\omega, \phi) \in \Gamma_{F^s}$, then either*

1. *F^s_ϕ has an A_1 singularity at ω or*
2. *F^s_ϕ has an A_2 singularity at ω.*

Proof The dense open subset $Q \cap M \cap P \subset C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ fulfils the necessary requirements where Q, M and P are defined in the proofs of Lemmas 8.3.1, 8.3.2 and 8.3.3 respectively. □

Corollary 8.3.5 *There exists a dense open subset of $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ such that for every element F, $F^s_\phi \in \Omega_3$ for every choice of (ϕ, s) in $S^1 \times I$.*

Assume that $F \in Q \cap M \cap P$ and that Γ_{F^s} has a double point singularity at (ω, ϕ). Then $\Gamma_{F^{s'}}$ is smooth everywhere for every s' in a deleted open

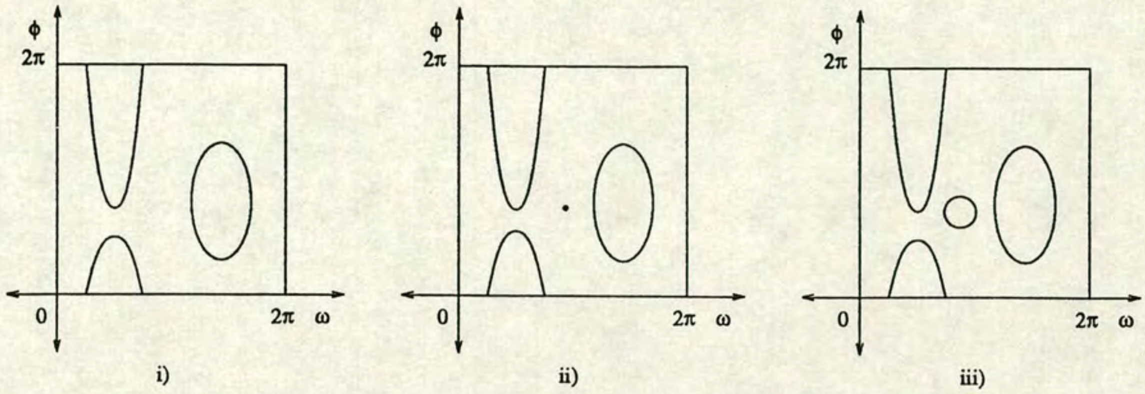

Figure 8.1: Elliptic Morse Reconstruction

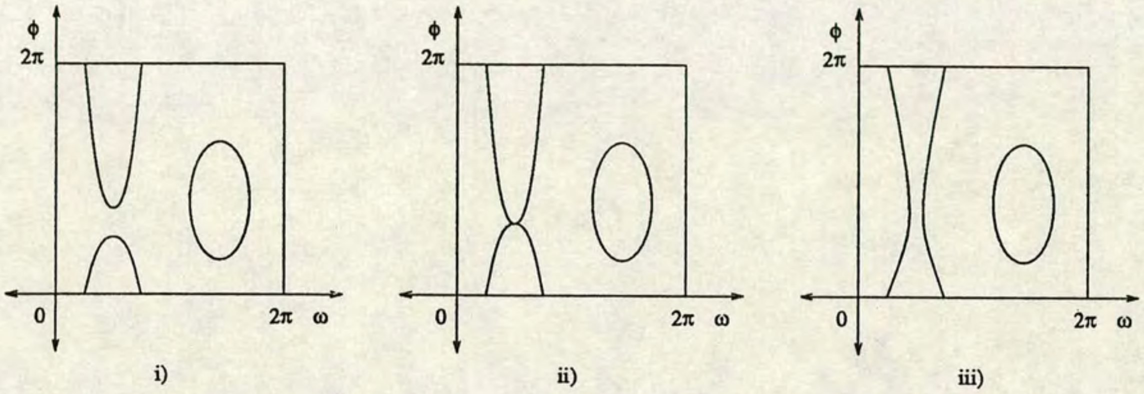

Figure 8.2: Hyperbolic Morse Reconstruction

neighbourhood of s in I. The isolated singularity of Γ_{F^s} illustrated in Figure 8.1 ii) either disappears under perturbation (with respect to s, see Figure 8.1 i)) or forms a disc in $S^1 \times S^1$, representing the formation of a maximum and a minimum on neighbouring fibres (see Figure 8.1 iii)). The passage from disc to point to empty set (or vice versa) is called elliptic Morse reconstruction.

Consider the behaviour of the isolated singularity of Γ_{F^s} illustrated in Figure 8.2 ii). For s' near s, either the critical graph separates (see Figure 8.2 i)) or we eliminate the A_2 singularity (see Figure 8.2 iii)). The passage from Figure 8.2 i) to Figure 8.2 ii) to Figure 8.2 iii) (or vice versa) is called hyperbolic Morse reconstruction.

Consider a family of closed curves in the torus, Γ^s, s in I, such that

1. the curves Γ^s are smooth for almost all s in I,
2. at critical values of s, the singularities of Γ^s are double point singularities (of the type $x^2 \pm y^2 = 0$),
3. these double point singularities do not have tangents which are horizontal (constant ϕ),
4. the singularities of Γ^s undergo Morse reconstruction at critical values of s and
5. if Γ^s is smooth at (ω, ϕ), then either Γ^s has a vertical tangent at (ω, ϕ) or Γ^s defines locally a Morse function $\phi = \phi(\omega)$.

Here we define s to be a critical value iff Γ^s is not smooth. Values of s which are not critical are called regular.

Definition 8.3.6 *We call such a family of curves an A_3-cobordism between Γ^0 and Γ^1. We call Γ^0 and Γ^1 A_3-cobordant.*

Lemma 8.3.7 *If $F \in Q \cap M \cap P \subset C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ then $\{\Gamma_{F^s} : s \in I\}$ is an A_3-cobordism.*

Proof If $F \in Q \cap M \cap P$ then the curves Γ_{F^s} are smooth for all s but a finite set in I. At critical values of s, the singularities of Γ_{F^s} are double point singularities which do not have horizontal tangents. (If $F \in Q \cap M \cap P$ and if Γ_{F^s} has a double point singularity at (ω, ϕ), then F^s_ϕ has an A_2 singularity at ω and so neither of the tangents to Γ_{F^s} at (ω, ϕ) is horizontal.) These singularities undergo Morse reconstruction.

If Γ_{F^s} is smooth at (ω', ϕ') then here either $\frac{\partial^2 F}{\partial\omega\partial\phi} \neq 0$ or $\frac{\partial^2 F}{\partial\omega^2} \neq 0$. If $\frac{\partial^2 F}{\partial\omega\partial\phi} = 0$ then $\frac{\partial^2 F}{\partial\omega^2} \neq 0$ and so here Γ_{F^s} has a vertical tangent and $F^s_{\phi'}$ has an A_1

singularity at ω'. If $\frac{\partial^2 F}{\partial\omega\partial\phi} \neq 0$ at (ω', ϕ') in Γ_{F^s} then locally, Γ_{F^s} is the graph of a smooth function $\phi = \phi(\omega)$. That is, the equation $\frac{\partial F^s}{\partial \omega} = 0$ defines implicitly a smooth function $\phi = \phi(\omega)$.

To relate the derivatives of ϕ to those of F^s expand the expression

$$\frac{\partial}{\partial\omega}(\frac{\partial F^s}{\partial\omega} = 0)$$

on Γ_{F^s} near (ω', ϕ'). Here we find that

$$\frac{\partial^2 F^s}{\partial\omega^2} + \frac{\partial^2 F^s}{\partial\omega\partial\phi}\frac{d\phi}{d\omega} = 0.$$

Hence, if Γ_{F^s} is smooth at (ω', ϕ') and here $\frac{\partial^2 F}{\partial\omega\partial\phi} \neq 0$ then $F^s_{\phi'}$ has a nondegenerate critical point at ω' iff ω' is a regular point of the function ϕ and $F^s_{\phi'}$ has a degenerate critical point at ω' iff ω' is a critical point of ϕ.

Differentiating once more we obtain

$$\frac{\partial^2}{\partial\omega^2}(\frac{\partial F^s}{\partial\omega} = 0)$$

on Γ_{F^s} near (ω', ϕ') and so here

$$\frac{\partial^3 F^s}{\partial\omega^3} + \frac{\partial^3 F^s}{\partial\omega^2\partial\phi}(\frac{d\phi}{d\omega}) + (\frac{\partial^3 F^s}{\partial\phi^2\partial\omega}(\frac{d\phi}{d\omega}) + \frac{\partial^3 F^s}{\partial\omega^2\partial\phi})(\frac{d\phi}{d\omega}) + \frac{\partial^2 F^s}{\partial\omega\partial\phi}(\frac{d^2\phi}{d\omega^2}) = 0.$$

Hence, if Γ_{F^s} is smooth at (ω', ϕ'), and here $\frac{\partial^2 F}{\partial\omega\partial\phi} \neq 0$ then $F^s_{\phi'}$ has an A_2 singularity at ω' iff ω' is a nondegenerate critical point of ϕ. Hence, if Γ_{F^s} is smooth at (ω', ϕ') then either Γ_{F^s} has a vertical tangent at (ω', ϕ'), or locally, the smooth function defined implicitly by Γ_{F^s} is Morse. □

8.4 From Cobordisms to Homotopies

A closed curve in the torus whose complement is labelled is a closed curve in the torus together with a labelling of the path components of its complement with plus and minus signs so that neighbouring path components always have opposite signs away from points of intersection. For our purpose, a closed curve may not be connected, but must have a finite number of path components. These curves arise as the critical graphs of parametrized generalized Morse functions on the torus.

Definition 8.4.1 *Let G be the set of all closed curves in the torus whose complement is labelled each one of which is smooth almost everywhere and is such that its transversal intersection with each horizontal (constant ϕ) is a non-zero even number of points and, at every point on the curve, either*

1. *the curve is smooth and has a vertical tangent or*
2. *the curve is smooth and defines locally a Morse function $\phi = \phi(\omega)$ or*
3. *the curve has a double point singularity which has no horizontal tangent.*

Note that neighbouring path components of the complement of a curve in G always have opposite signs at smooth points on the curve.

An A_3-cobordism in G is an A_3-cobordism whose curves are in G such that the labelling of the path components of the complements of the curves changes "continuously".

Lemma 8.4.2 *1. Each closed curve Γ in G is the critical graph Γ_f for some parametrized generalized Morse function f in $C^\infty(S^1 \times S^1, \mathbf{R})$ such that $\frac{\partial f}{\partial \omega}$ is strictly positive on positive components of $S^1 \times S^1 \backslash \Gamma$ and such that $\frac{\partial f}{\partial \omega}$ is strictly negative on negative components of $S^1 \times S^1 \backslash \Gamma$.*

2. Each A_3-cobordism in G can be generated by a homotopy in the space of parametrized generalized Morse functions on the torus.

Proof 1. Choose Γ in G. Find a function g in $C^\infty(S^1 \times S^1, \mathbf{R})$ such that $g(\omega, \phi) = 0$ defines Γ, g is Morse with respect to ω and regular with respect to ϕ where Γ is smooth with horizontal tangents, g is regular with respect to ω elsewhere where Γ is smooth, g is Morse with respect to ω and ϕ at singularities of Γ, g is strictly positive on positive components of $S^1 \times S^1 \backslash \Gamma$ and g is strictly negative on negative components of $S^1 \times S^1 \backslash \Gamma$.

Adjust g off Γ so that

$$\int_0^{2\pi} g(\omega, \phi)\, d\phi = \int_0^{2\pi} g(\omega, \phi)\, d\omega = 0$$

everywhere.

Let

$$f(\omega, \phi) = \int_0^{\omega} g(\omega', \phi)\, d\omega' = f_\phi(\omega)$$

where ω and ϕ are in $\mathbf{R}/2\pi\mathbf{Z}$. Then $f_\phi \in \Omega_3$, for every ϕ in $\mathbf{R}/2\pi\mathbf{Z}$ and $\Gamma_f = \Gamma$.

If f_0 and f_1 in $C^\infty(S^1 \times S^1, \mathbf{R})$ are two such parametrized generalized Morse functions then so is $tf_0 + (1-t)f_1$, for every t in I.

2. Let $\{\Gamma^s : s \in I\}$ be an A_3-cobordism between Γ^0 and Γ^1. Choose A in $C^\infty(S^1 \times S^1 \times I, \mathbf{R})$ so that, if $A^s(\omega, \phi) = A(\omega, \phi, s)$ everywhere then

$$\{A^s : s \in I\}$$

is a family of functions each one of which obeys the conditions above. Define

$$\begin{array}{rcrcl} F & : & S^1 \times S^1 \times I & \to & \mathbf{R} \\ & & (\omega, \phi, s) & \mapsto & \int_0^\omega A^s(\omega', \phi)\, d\omega'. \end{array} \qquad \text{by}$$

Then $\Gamma_{F^s} = \Gamma^s$ for each s in I. □

8.5 Equivalence Classes of Closed Curves in the Torus

Lemma 8.5.1 *The relation of A_3-cobordism is an equivalence relation on G.*

Proof It is immediate from the definition of A_3-cobordism that the relation is symmetric, reflexive and transitive. □

Lemma 8.5.2 *Every element in G is A_3-cobordant in G to a curve consisting of a finite number of kidneys (defined below) enclosing positive regions.*

Proof Choose Γ in G, then Γ is A_3-cobordant to some $\tilde{\Gamma}$ in G, whose points with horizontal tangents are all at distinct critical values of ϕ (see Figure 8.3) and are finite in number. Choose one non-critical value ϕ_i between each adjacent pair of critical values in S^1. The closed curve $\phi = \phi_i$ intersects the positive regions in segments. Choose one point in each of these segments (see Figure 8.4) and perform hyperbolic Morse reconstruction at each of these points to get a curve consisting of components with two or four horizontal tangents which we call discs or "kidneys", respectively. In each strip between successive critical values there is at most one kidney (see Figure 8.5).

The new curve fails to be an element of G as it doesn't intersect transversally the non-critical horizontals used in its construction (see Figure 8.5). To overcome this, introduce small discs over the non-critical horizontals (elliptic Morse reconstruction) while simultaneously performing the hyperbolic Morse reconstructions described above (see Figure 8.6).

The resulting curve is an element of G, A_3-cobordant in G to the original. It consists of discs and kidneys enclosing positive regions. The discs may be eliminated by elliptic Morse reconstruction. Pairs of kidneys in reverse orientation may be eliminated by hyperbolic Morse reconstruction followed by elliptic Morse reconstruction. The constraint that each element in G is such that its transversal intersection with each horizontal is a non-zero even number of points means that, if all the discs are eliminated, then some kidneys remain (see Figures 8.6 and 8.7). Moreover, while eliminating discs, it may be necessary

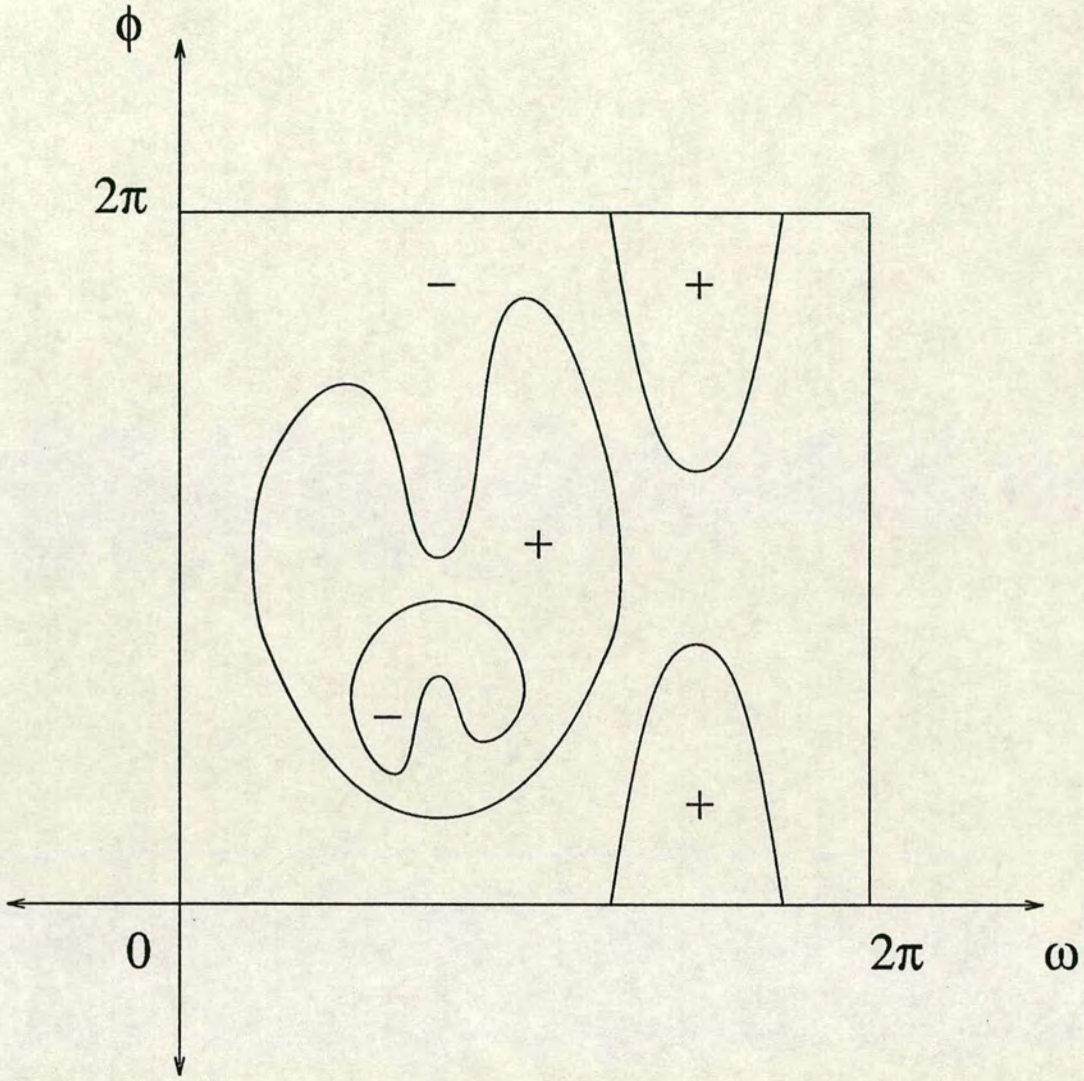

Figure 8.3: The Curve $\tilde{\Gamma}$

to create pairs of kidneys in reverse orientation. □

Denote by $[G]$ the set of equivalence classes of G and denote by $[\Gamma]$ the equivalence class of Γ in G. We define a binary operation on $[G]$ with respect to which $[G]$ is a group.

Multiplication

Outlined below is a method for multiplying equivalence classes of G. Let Γ^1 and Γ^2 be elements of G, then a representative curve for the product $[\Gamma^1][\Gamma^2]$ may be visualized by "stacking" the diagrams of the curves Γ^1 and Γ^2. By Lemma 8.5.2,

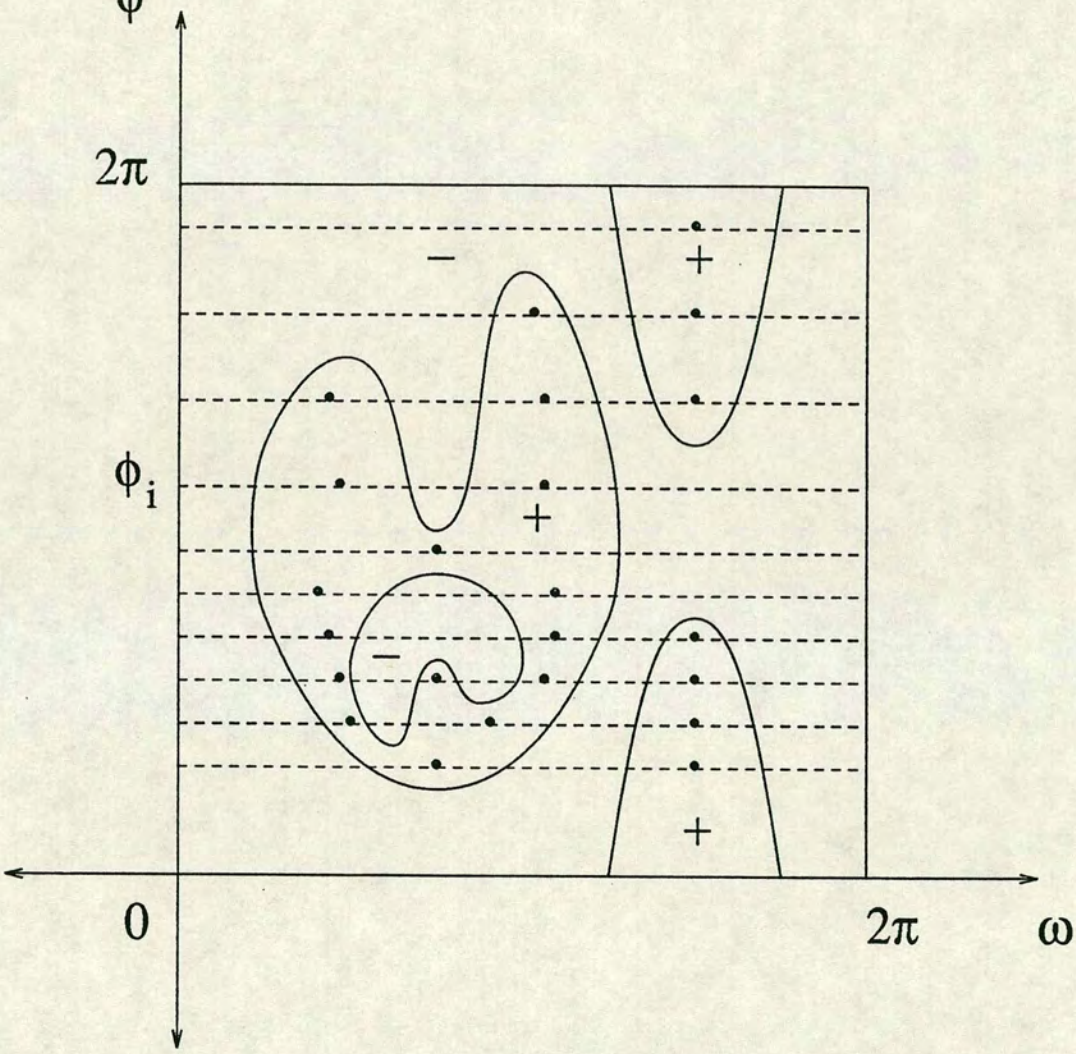

Figure 8.4: Non-Critical Horizontals

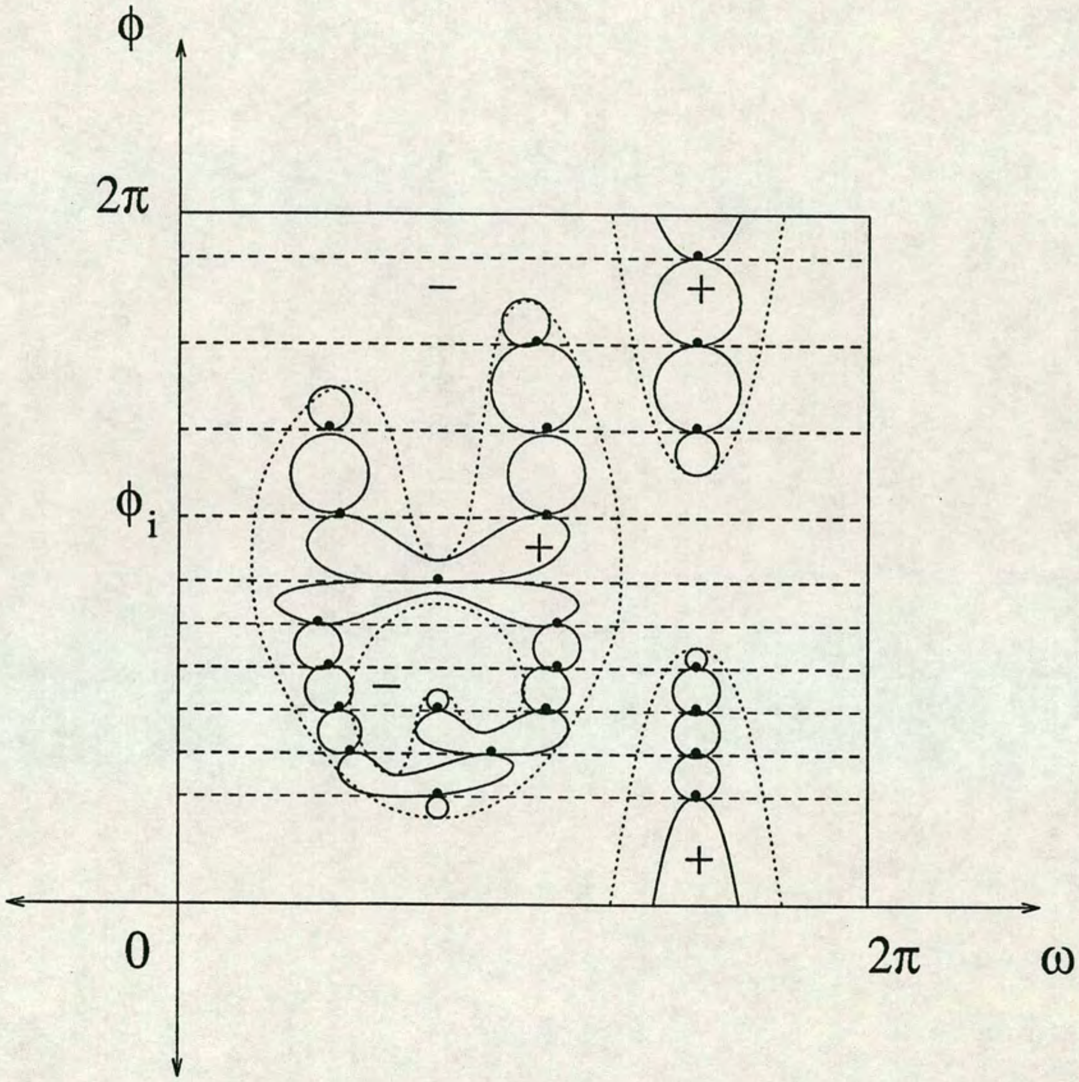

Figure 8.5: Hyperbolic Morse Reconstruction

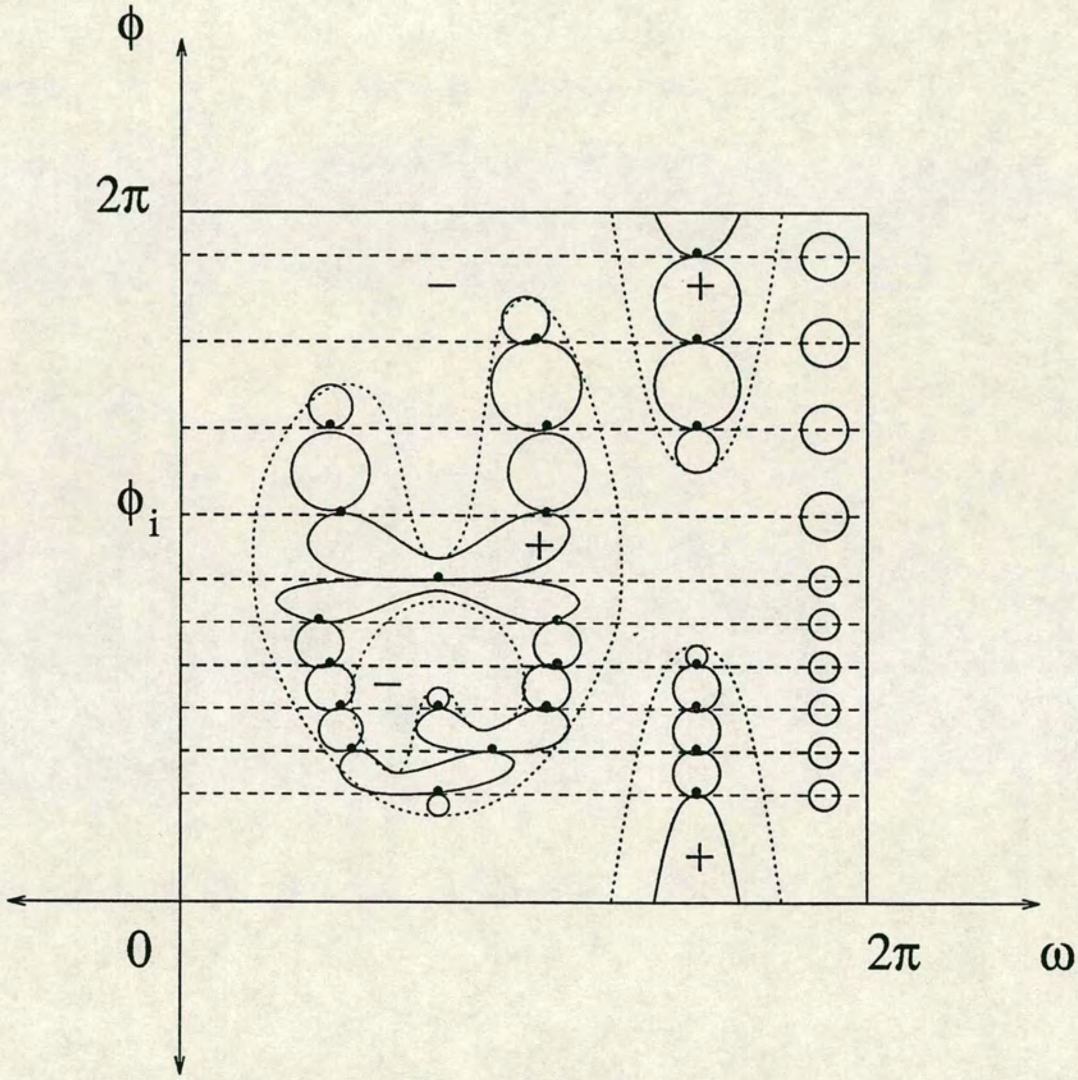

Figure 8.6: Elliptic Morse Reconstruction

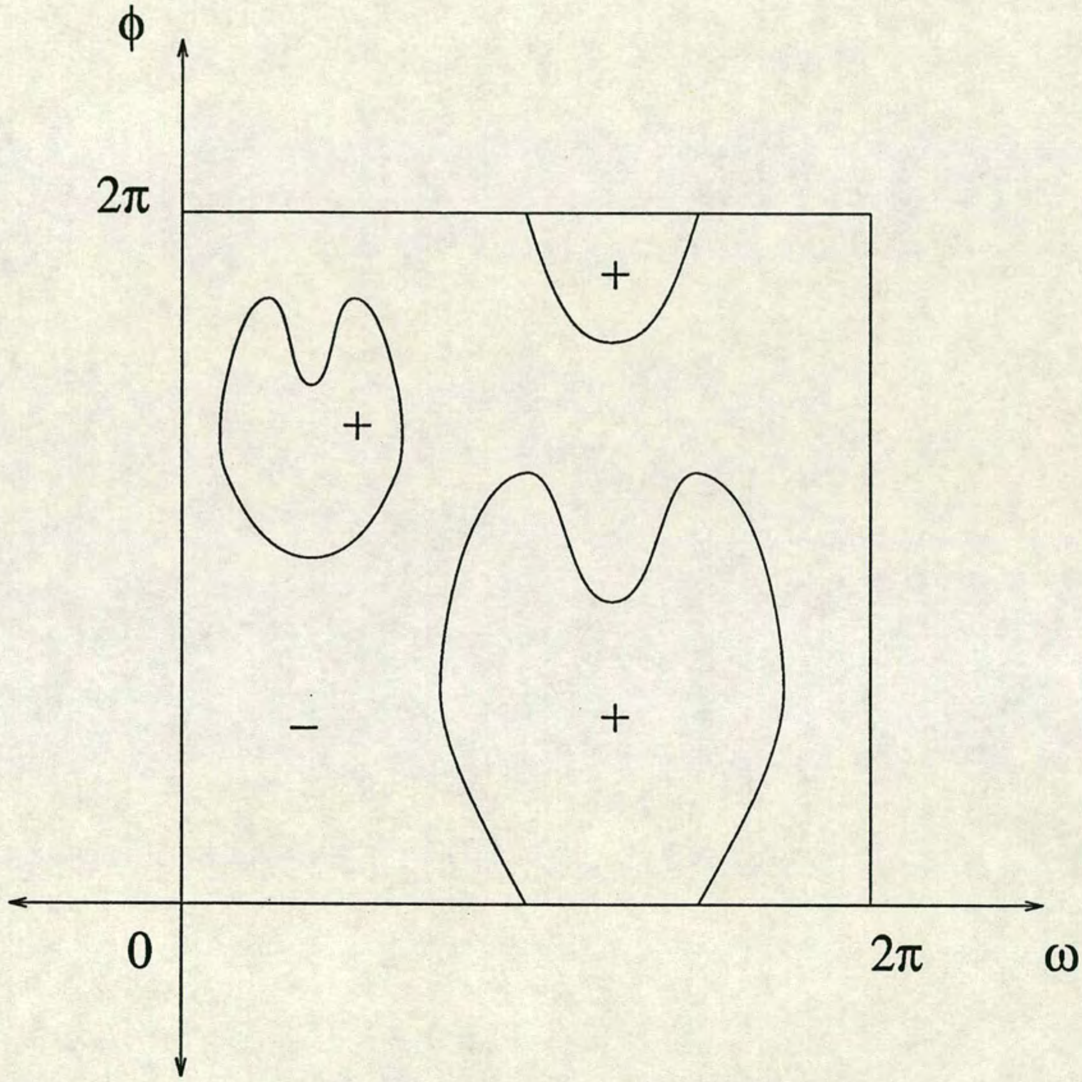

Figure 8.7: Eliminating Kidneys and Discs

Γ^1 and Γ^2 are A_3-cobordant in G to some curves $\Gamma^{1'}$ and $\Gamma^{2'}$ respectively, consisting of kidneys enclosing positive regions.. Before stacking the diagrams of the curves, perform A_3-cobordisms in G with initial curves $\Gamma^{1'}$ and $\Gamma^{2'}$ as suggested in Figure 8.8, squeezing each curve into the "centre" of the torus, that is, pulling it away from the horizontal $\phi = 0(\text{mod}2\pi)$ (see Figure 8.8 i) and ii)). The horizontal $\phi = 0(\text{mod}2\pi)$ is now contained in a path component with negative sign. A disc enclosing a positive region is added over each horizontal $\phi = 0(\text{mod}2\pi)$ to ensure that the resulting curves are in G. The curve A_3-cobordant to $\Gamma^{1'}$ is stacked above the curve A_3-cobordant to $\Gamma^{2'}$, joined by matching the discs over the horizontal $\phi = 0(\text{mod}2\pi)$. The resulting curve is in G (see Figure 8.8 iii)) and is defined up to equivalence.

Definition 8.5.3 *Given $[\Gamma^1]$ and $[\Gamma^2]$ in $[G]$, the product $[\Gamma^1][\Gamma^2]$ is defined to be the equivalence class of a curve in G constructed from representative curves Γ^1 and Γ^2 by the stacking method described above.*

Clearly, multiplication in $[G]$ is associative.

Identity Element

Consider the map

$$(\omega, \phi) \mapsto sin(\omega).$$

The critical graph $\Gamma_{sin(\omega)}$ has the form of two vertical lines (see Figure 8.9) and $[\Gamma_{sin(\omega)}][\Gamma] = [\Gamma] = [\Gamma][\Gamma_{sin(\omega)}]$, for any Γ in G. Hence $[\Gamma_{sin(\omega)}]$ is an identity element with respect to multiplication in $[G]$.

Inversion

Let Γ be an element of G, then by Lemma 8.4.2, $\Gamma = \Gamma_f$ for some parametrized generalized Morse function f. Define $\overline{f}$ in $C^\infty(S^1 \times S^1, \mathbf{R})$ by

$$\overline{f}(\omega, \phi) = f(\omega, 2\pi - \phi).$$

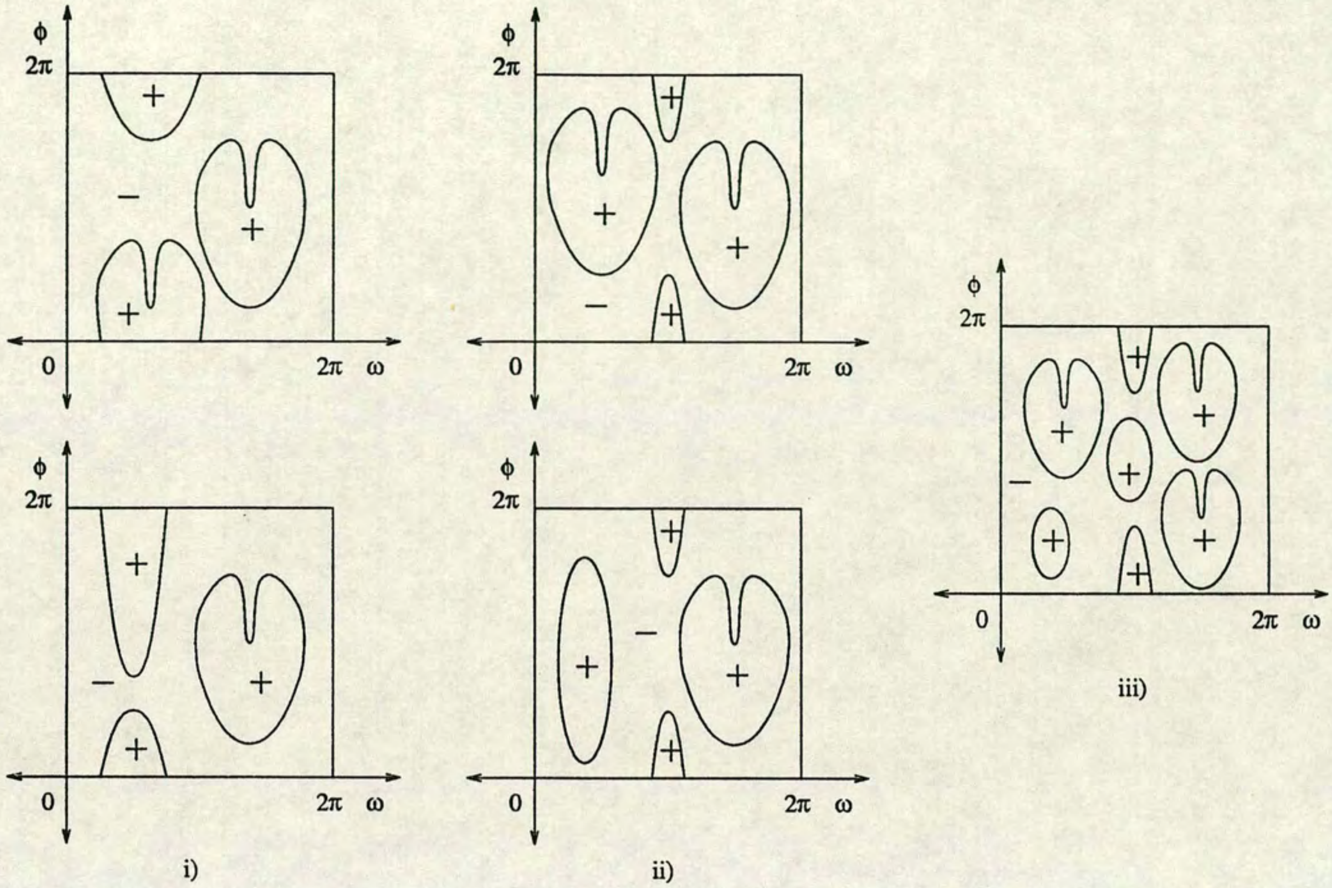

Figure 8.8: Multiplying Equivalence Classes of G

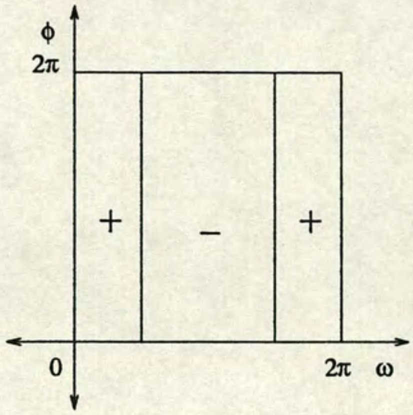

Figure 8.9: $\Gamma_{sin(\omega)}$

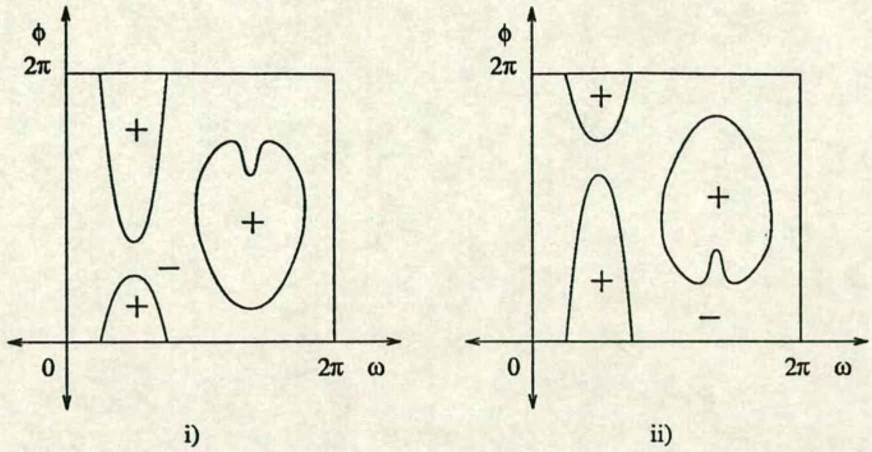

Figure 8.10: A Curve and its Inverse

Then $[\Gamma_f][\Gamma_{\overline{f}}] = [\Gamma_{\overline{f}}][\Gamma_f] = [\Gamma_{sin(\omega)}]$, the identity element defined above (see Figure 8.10). Note that $[\Gamma_{\overline{f}}]$ depends only upon $[\Gamma]$ in $[G]$ and not upon the choice of f.

Definition 8.5.4 *The inverse of* $[\Gamma]$ *in* $[G]$ *is denoted by* $[\Gamma]^{-1}$ *and is equal to* $[\Gamma_{\overline{f}}]$ *where* $\Gamma = \Gamma_f$.

Corollary 8.5.5 *The equivalence classes of* G *form a group with respect to the associative operation of multiplication described above.*

Theorem 8.5.6 *The group* $[G]$ *is isomorphic to* $\mathbb{Z}$.

Proof We define a map

$$Index : G \to \mathbb{Z}$$

which is invariant under A_3-cobordism and is an isomorphism on equivalence classes of G. Consider the kidney enclosing a positive region, illustrated in Figure 8.11. The kidney may enclose further positive and negative regions (see Figure 8.12). Consider the points on the kidney with horizontal tangents. Viewed locally as the graph of a real-valued function, the kidney has two maxima and two minima (see Figure 8.12). If the curve is convex (concave) with respect to its positive interior at a maximum or a minimum the extremum is denoted convex (concave). If the curve is convex (concave) with respect to its negative interior at a maximum or a minimum the extremum is denoted concave (convex).

We define the map

$$\begin{array}{rcccl} Index & : & G & \rightarrow & \mathbb{Z} \\ & & \Gamma & \mapsto & \#\text{convex maxima} - \#\text{convex minima}. \end{array} \quad \text{by}$$

Note that the index of a curve is invariant under elliptic or hyperbolic Morse reconstruction. Hence $Index$ is invariant under A_3-cobordism. Hence $Index$ is constant on equivalence classes.

By the proof of Lemma 8.5.2, every element in G is A_3-cobordant in G to a curve consisting of a finite disjoint union of kidneys enclosing positive regions, all of the same orientation, or to a curve consisting entirely of discs. Hence each equivalence class is equal to a power of the generator which is represented by the kidney in Figure 8.11. Since multiplication in $[G]$ is effected by stacking curves, it follows that $Index$ induces a group homomorphism from $[G]$ onto $\mathbb{Z}$. If Γ is in G and $Index(\Gamma) = 0$ then Γ is A_3-cobordant in G to a curve in G consisting entirely of discs. Hence $Index(\Gamma) = 0$ implies that Γ is A_3-cobordant in G to $\Gamma_{sin(\omega)}$ (see Figure 8.9). Hence $Index(\Gamma) = 0$ if and only if Γ is A_3-cobordant in G to $\Gamma_{sin(\omega)}$. Hence $Index$ induces a group isomorphism from $[G]$ onto $\mathbb{Z}$. $\square$

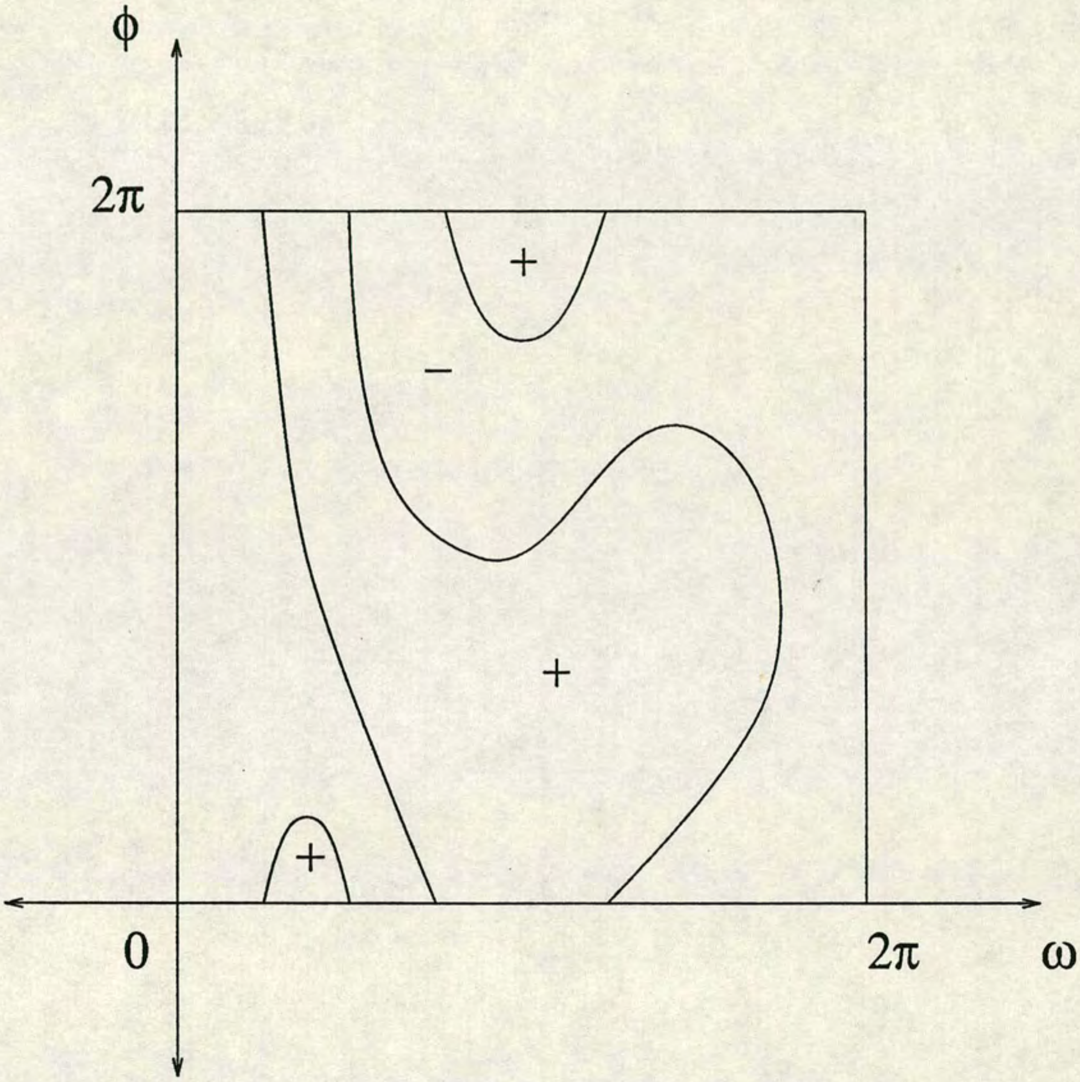

Figure 8.11: Kidney

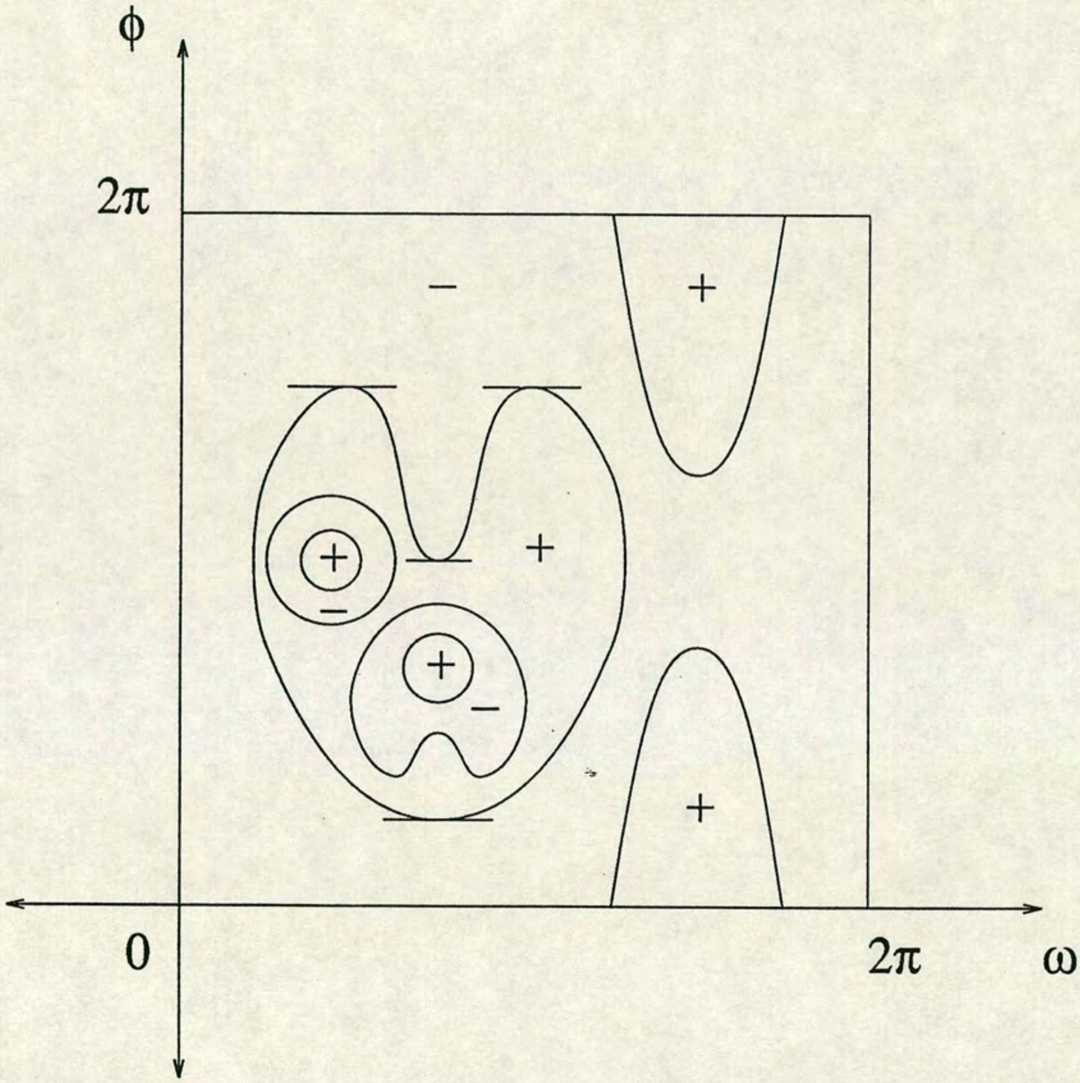

Figure 8.12: Embedded Kidneys and Discs

8.6 The Fundamental Group of Ω_3

Theorem 8.6.1 *The group $[G]$ is isomorphic to $\pi_1(\Omega_3, \omega \mapsto \cos(\omega))$.*

Proof Let

$$L_0 \; : \; \mathbf{R}/2\pi\mathbf{Z} \;\to\; \Omega_3$$

be a continuous loop such that $L_0(0) = L_0(2\pi) = (\omega \mapsto \cos(\omega))$ and such that the induced function on the torus given by

$$\begin{array}{rccl} f(L_0) & : & S^1 \times S^1 & \to & \mathbf{R} \\ & & (\omega, \phi) & \mapsto & L_0(\phi)(\omega) \end{array}$$

is a parametrized generalized Morse function.

It might be that the critical graph of $f(L_0)$ is not in G. In this case, let

$$L \; : \; \mathbf{R}/2\pi\mathbf{Z} \times [0,1] \;\to\; \Omega_3$$

be a homotopy such that $L(\phi, 0) = L_0(\phi)$, $L(0,t) = L(2\pi, t) = (\omega \mapsto \cos(\omega))$ for all t in $[0,1]$ and such that the critical graph $\Gamma_{f(L_1)}$ is in G (where $L(\phi, t) = L_t(\phi)$, for all t in $[0,1]$ and all ϕ in $\mathbf{R}/2\pi\mathbf{Z}$). L is constructed by gradually eliminating A_2 singularities on fibres in the torus until only a finite number remain.

Suppose

$$M \; : \; \mathbf{R}/2\pi\mathbf{Z} \times [0,1] \;\to\; \Omega_3$$

is another homotopy such that $M(\phi, 0) = L_0(\phi)$, $M(0,t) = M(2\pi, t) = (\omega \mapsto \cos(\omega))$ for all t in $[0,1]$ and such that the critical graph $\Gamma_{f(M_1)}$ is in G (where $M(\phi,t) = M_t(\phi)$, for all t in $[0,1]$ and all ϕ in $\mathbf{R}/2\pi\mathbf{Z}$). Perturb the path from M_1 to L_1 via L_0 to produce a homotopy

$$N \; : \; \mathbf{R}/2\pi\mathbf{Z} \times [0,1] \;\to\; \Omega_3$$

such that $N(\phi, 0) = M_1(\phi)$, $N(\phi, 1) = L_1(\phi)$, $N(0, t) = N(2\pi, t) = (\omega \mapsto \cos(\omega))$ for all t in $[0, 1]$ and such that the set $\{\Gamma_{f(N_t)} : t \in [0, 1]\}$ is an A_3-cobordism in G. N is constructed by gradually eliminating A_2 singularities on fibres in the torus until only a finite number remain in $S^1 \times S^1 \times [0, 1]$. It follows that $\Gamma_{f(L_1)}$ and $\Gamma_{f(M_1)}$ must be A_3-cobordant in G.

Hence we can define a map

$$\begin{array}{rcrcl} Curve & : & \pi_1(\Omega_3, \omega \mapsto \cos(\omega)) & \to & [G] \qquad \text{by} \\ & & [L_0] & \mapsto & [\Gamma_{f(L_1)}] \end{array}$$

where $[L_0]$ is the homotopy class of L_0. The map $Curve$ is a group homomorphism as multiplication in $\pi_1(\Omega_3, \omega \mapsto \cos(\omega))$ is effected by composing paths and multiplication in $[G]$ is effected by stacking curves.

By Lemma 8.4.2, each closed curve Γ in G is the critical graph Γ_f for some parametrized generalized Morse function f in $C^\infty(S^1 \times S^1, \mathbf{R})$. Choose $\tilde{\Gamma}$ in G, A_3-cobordant in G to Γ, such that $\tilde{\Gamma}$ is the critical graph for some parametrized generalized Morse function $\tilde{f}$ in $C^\infty(S^1 \times S^1, \mathbf{R})$ such that $\tilde{f}(\omega, 0) = \tilde{f}(\omega, 2\pi) = \cos(\omega)$ for all ω in $\mathbf{R}/2\pi\mathbf{Z}$. Then the equivalence class in $\pi_1(\Omega_3, \omega \mapsto \cos(\omega))$ of the loop

$$\begin{array}{rcrcl} L(\tilde{f}) & : & \mathbf{R}/2\pi\mathbf{Z} & \to & \Omega_3 \quad \text{given by} \\ & & \phi & \mapsto & \tilde{f}_\phi \end{array}$$

maps to $[\tilde{\Gamma}]$. Hence $[\Gamma]$ is in the image of the map $Curve$. Hence the map $Curve$ is an epimorphism.

Suppose $Curve([L_0])$ is the identity element in G. Then L_0 is homotopic, relative to the set $\{[0]\}$ in $\mathbf{R}/2\pi\mathbf{Z}$, to a loop L_1 whose critical graph $\Gamma_{f(L_1)}$ has index zero. Hence L_0 is homotopic to the constant loop

$$\begin{array}{rcrcl} L_{\cos(\omega)} & : & \mathbf{R}/2\pi\mathbf{Z} & \to & \Omega_3 \\ & & L_{\cos(\omega)}(\phi) & \mapsto & (\omega \mapsto \cos(\omega)). \end{array}$$

Hence the map $Curve$ is an monomorphism. Hence the map $Curve$ is a group isomorphism. $\square$

Theorem 8.6.2 *The fundamental group $\pi_1(\Omega_3, \omega \mapsto \cos(\omega))$ is isomorphic to the group of integers $\mathbb{Z}$.*

Proof By Theorem 8.5.6 and Theorem 8.6.1, the map

$$Index \circ Curve \ : \ \pi_1(\Omega_3, \omega \mapsto \cos(\omega)) \ \rightarrow \ [G] \rightarrow \mathbb{Z}$$

is a group isomorphism (see [4]). $\square$

Recall from §8.1 that Ω_3 is path connected. Hence the calculation of the fundamental group is independent of the chosen base point.

Corollary 8.6.3 *Let f in $C^\infty(S^1 \times S^1, \mathbb{R})$ be a parametrized generalized Morse function whose labelled critical graph is in G. Then the map*

$$\phi \mapsto f_\phi \ : \ S^1 \ \rightarrow \ \Omega_3$$

1. *is null homotopic iff $Index(\Gamma_f) = 0$ and*
2. *represents a generator of $\pi_1(\Omega_3)$ iff $Index(\Gamma_f) = \pm 1$.*

Chapter 9

Simple Singularities on Compact Principal S^1-Bundles

Throughout this chapter $\pi : E \to B$ denotes a smooth principal S^1- bundle with total space E and base space B of dimension m, $0 \leq m < \infty$. In Corollary 4.5.3 it was proved that the set of all smooth real-valued functions on E whose singularities on fibres are only of types $A_1, \ldots, A_{m+1}$ is open and dense in the Whitney C^∞ topology. In this chapter, for E compact and $m \geq 2$, we prove the existence of a smooth real-valued function on E whose singularities on fibres are only of types $A_1, \ldots, A_{[\frac{m}{2}]+1}$. In terms of existence, this is an improvement on Corollary 4.5.3 when E is compact.

9.1 Equivariance

Consider the map

$$(\exp(i\omega), f) \mapsto \exp(i\omega)f \quad : \quad S^1 \times \Omega_k \quad \to \quad \Omega_k$$

where

$$(\exp(i\omega)f)(\exp(i\theta)) = f(\exp(i\omega)\exp(i\theta))$$

for every ω and θ in $\mathbf{R}/2\pi\mathbf{Z}$ and f in Ω_k. (For a definition of Ω_k, $k \geq 2$, see §7.1.)

Definition 9.1.1 *If g in $C^0(E, \Omega_k)$, $k \geq 2$, is such that*

$$\exp(i\omega)(g(e)) = g(\exp(i\omega)e)$$

for every e in E and for every $\exp(i\omega)$ in S^1 then we call g equivariant with respect to the action of S^1.

For fixed $k \geq 2$, let

$$P = \{f \in C^0(E, \mathbb{R}) \text{ such that } f \in \Omega_k \text{ on fibres }\},$$

and let

$$Q = \{g \in C^0(E, \Omega_k) \text{ such that } g \text{ is equivariant with respect to the action of } S^1\}.$$

Define

$$\begin{array}{rcccl} G & : & P & \to & Q \\ & & f & \mapsto & G(f) \end{array} \quad \text{by}$$

where

$$\begin{array}{rcccl} G(f) & : & E & \to & \Omega_k \\ & & e & \mapsto & \{\exp(i\omega) \mapsto f(\exp(i\omega)e)\}. \end{array} \quad \text{is defined by}$$

Note that $G(f)$ is equivariant with respect to the action of S^1.

9.2 A Theorem by Vasil'ev

Some standard homotopy theory is used in this section (see [42] for details).

Definition 9.2.1 *Let X and Y be topological spaces. A continuous map $f : X \to Y$ is called an n-equivalence for $n \geq 1$ if f induces a one-to-one correspondence between the path components of X and of Y and if for every x in X, the induced map on homotopy groups*

$$f_* : \pi_q(X, x) \to \pi_q(Y, f(x))$$

is an isomorphism for $0 \le q \le n-1$ and an epimorphism for $q = n$. A map $f : X \to Y$ is called a weak homotopy equivalence if f is an n-equivalence for every $n \ge 1$ (see [42, page 404]).

We point out that a homotopy equivalence is a weak homotopy equivalence but the converse is not true. However, a map between CW-complexes is a weak homotopy equivalence iff it is a homotopy equivalence [42, page 405].

Notation 9.2.2 *Denote $C^0(S^1, S^n)$, the free loop space on S^n, by $L(S^n)$, $n \ge 1$.*

As in §7.3, identify $\mathbf{R}/2\pi\mathbf{Z}$ with S^1 by the canonical map

$$\omega \mapsto \exp(i\omega)$$

and consider a function in Ω as a smooth function on $\mathbf{R}$, periodic, with period 2π.

Define the map

$$p_n : \Omega_n \rightarrow L(S^{n-1}) \text{ by}$$
$$p_n(f)(\omega) = \frac{(f^{[1]}(\omega), \ldots, f^{[n]}(\omega))}{\sqrt{\sum_{i=1}^{n} \left(f^{[i]}(\omega)\right)^2}}$$

where $f \in \Omega_n$,and $\omega \in \mathbf{R}/2\pi\mathbf{Z}$. Ignoring the constant term, $p_n(f)(\omega)$ is a scaled n-jet of f at ω.

Theorem 9.2.3 *The map $p_n : \Omega_n \to L(S^{n-1})$ is a weak homotopy equivalence for every $n \ge 3$ (see Vasil'ev[47]).*

Corollary 9.2.4 *If*

$$i_n : \Omega_n \rightarrow \Omega_{n+1}$$

is the inclusion map, $n \ge 2$, then i_n induces the zero map on all homotopy groups.

Proof The inclusion

$$\begin{array}{rcrcl} j_n & : & S^{n-1} & \to & S^n \\ & & \underline{r} & \mapsto & (\underline{r}, 0) \end{array}$$

where $\underline{r} \in \mathbf{R}^n$ and $(\underline{r}, 0) \in \mathbf{R}^{n+1}$ induces a canonical map

$$\tilde{j_n} \ : \ L(S^{n-1}) \ \to \ L(S^n)$$

for every $n \geq 2$. Consider the following map diagram (which is homotopy commutative).

$$\begin{tikzcd} \Omega_n \arrow[r, "p_n"] \arrow[d, "i_n"'] & L(S^{n-1}) \arrow[d, "\tilde{j_n}"] \\ \Omega_{n+1} \arrow[r, "p_{n+1}"] & L(S^n) \end{tikzcd}$$

By Vasil'ev's Theorem 9.2.3, p_{n+1} is a weak homotopy equivalence for every $n \geq 2$. As $\tilde{j_n} \circ p_n$ is homotopic to $p_{n+1} \circ i_n$ and $\tilde{j_n}$ induces the zero map on all homotopy groups it follows that i_n induces the zero map on all homotopy groups for every $n \geq 2$. □

9.3 Simple Singularities on Hopf Bundles

The main result of this section is the following theorem.

Theorem 9.3.1 *For every $n \geq 0$ let*

$$\pi^n \ : \ S^{2n+1} \ \to \ \mathbf{CP}^n$$

be the Hopf map (see A.4). Then there exists some f_n in $C^\infty(S^{2n+1}, \mathbf{R})$ which is in Ω_{n+2} on fibres.

We require a number of preliminary results before we can prove Theorem 9.3.1. Give both $\mathbf{R}^n$ and $\mathbf{C}^n$ the usual Euclidean metric and topology and give each subset the subspace topology (see [3, page 28]). As in §6.3, denote by D^n the closed unit ball in $\mathbf{R}^n$, denote by $(D^n)^0$ its interior and by ∂D^n or S^{n-1} its boundary, n in $\mathbf{N}$. Identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ in the canonical way. This induces an identification of S^{2n-1} with

$$\{(z_1, \ldots, z_n) \in \mathbf{C}^n \ : \ \Sigma_{i=1}^n |z_i|^2 = 1\}.$$

Denote by $S^1 \times D^{2n}/_\sim$ the identification space formed by identifying elements of the same equivalence class with respect to the equivalence relation

$$(\exp(i\omega), \underline{z}) \sim (\exp(i\omega'), \underline{z'}) \text{ iff}$$

1. $\underline{z} = \underline{z'}$ and $\exp(i\omega) = \exp(i\omega')$ or
2. $|\underline{z}| = 1$ and $\exp(i\omega)\underline{z} = \exp(i\omega')\underline{z'}$.

As S^1 and D^{2n} are compact and Hausdorff, so is $S^1 \times D^{2n}/_\sim$. We show that $S^1 \times D^{2n}/_\sim$ and S^{2n+1} are homeomorphic whenever $n \in \mathbf{N}$.

Lemma 9.3.2 *A one-to-one, onto and continuous map from a compact space to a Hausdorff space is a homeomorphism.*

For proof see [3, page 48].

Lemma 9.3.3 *The spaces $S^1 \times D^{2n}/_\sim$ and S^{2n+1} are homeomorphic, for every n in $\mathbf{N}$.*

Proof Consider the map

$$\begin{array}{rccl} M_n : & S^1 \times D^{2n} & \rightarrow & S^{2n+1} \\ & (\exp(i\omega), \underline{z}) & \mapsto & (\exp(i\omega)(1 - |\underline{z}|^2)^{\frac{1}{2}}, \exp(i\omega)\underline{z}). \end{array} \quad \text{defined by}$$

The map M_n is continuous, onto and constant on equivalence classes in $S^1 \times D^{2n}$. Hence M_n induces a map

$$\begin{array}{rcrcl} \tilde{M_n} & : & S^1 \times D^{2n}/_\sim & \to & S^{2n+1} \\ & & [(\exp(i\omega), \underline{z})] & \mapsto & M_n(\exp(i\omega), \underline{z}). \end{array} \quad \text{given by}$$

The map $\tilde{M_n}$ is a continuous bijection. By Lemma 9.3.2, it is a homeomorphism. □

Next we give a technical smoothing lemma adapted from [37, Lemma 4.1].

Lemma 9.3.4 *Let U be an open subset of $\mathbf{R}^m$. Let A be a compact subset of the open set $V \subset \mathbf{R}^m$ such that the closure $\overline{V}$ is contained in U, and $\overline{V}$ is compact. Let $f : U \times S^1 \to \mathbf{R}$ be continuous and smooth on each fibre $\{\underline{x}\} \times S^1$, $\underline{x}$ in U. Let δ be a positive number. Let $f_{\underline{x}}(t)$ equal $f(\underline{x}, \exp(it))$ where $t \in \mathbf{R}$. Then there is a map $\tilde{f} : U \times S^1 \to \mathbf{R}$ such that*

1. *$\tilde{f}$ is of class C^∞ on a neighbourhood of $A \times S^1$,*

2. *$\tilde{f}$ equals f outside $V \times S^1$,*

3. *for every $\underline{x}$ in U and every t in $\mathbf{R}$*

$$\left|\tilde{f}_{\underline{x}}(t) - f_{\underline{x}}(t)\right| < \delta$$

   and for every $\underline{x}$ in U

$$\left|\frac{d^l \tilde{f}_{\underline{x}}(t)}{dt^l} - \frac{d^l f_{\underline{x}}(t)}{dt^l}\right| < \delta,$$

   everywhere, $1 \leq l \leq k$, for any chosen $k \geq 0$ and

4. *$\tilde{f}$ is of class C^p on any open set on which f is of class C^p, $0 \leq p \leq \infty$.*

Proof Let W be an open set containing A such that $\overline{W} \subset V$. Let Ψ be a C^∞ function on $\mathbf{R}^m$ which equals 1 in a neighbourhood of A and equals zero outside W. Define

$$\begin{array}{rcrcl} g & : & U \times S^1 & \to & \mathbf{R} \\ & & (\underline{x}, \exp(it)) & \mapsto & \Psi(\underline{x}) \cdot f(\underline{x}, \exp(it)). \end{array} \quad \text{by}$$

Extend g to $\mathbf{R}^m \times S^1$ by letting it equal zero outside $\overline{W} \times S^1$, then g is continuous and smooth on fibres.

For ϵ in $\mathbf{R}^+$, the closed ϵ-cube in $\mathbf{R}^m$, $C(\epsilon)$, is equal to

$$\{\underline{x} \in \mathbf{R}^m \ : \ |x_i| \leq \epsilon, 1 \leq i \leq m\}.$$

Let $\Phi(\underline{x})$ be a smooth function on $\mathbf{R}^m$ which is positive on the interior of $C(\epsilon)$ and zero elsewhere. Here ϵ is yet to be chosen. Assume that

$$\int_{C(\epsilon)} \Phi(\underline{x}) d\underline{x} = 1.$$

Define the function

$$\begin{array}{rrcll} h & : & U \times S^1 & \rightarrow & \mathbf{R} & \text{by} \\ & & (\underline{x}, \exp(it)) & \mapsto & \int_{C(\epsilon)} \Phi(\underline{y}) g(\underline{x} + \underline{y}, \exp(it)) d\underline{y}. & \end{array}$$

Choose $\sqrt{m}\epsilon$ less than the distance from W to the complement of V. Then $h(\underline{x}, \exp(it)) = 0$ for $\underline{x}$ outside V.

Define the function

$$\begin{array}{rrcll} \tilde{f} & : & U \times S^1 & \rightarrow & \mathbf{R} & \text{by} \\ & & (\underline{x}, \exp(it)) & \mapsto & f(\underline{x}, \exp(it))(1 - \Psi(\underline{x})) + h(\underline{x}, \exp(it)). & \end{array}$$

Since $\Psi(\underline{x}) = 0$ and $h(\underline{x}, \exp(it)) = 0$ for $\underline{x}$ outside V the second requirement of the lemma is satisfied. Now

$$\begin{aligned} h(\underline{x}, \exp(it)) &= \int_{C(\epsilon)} \Phi(\underline{y}) g(\underline{x} + \underline{y}, \exp(it)) d\underline{y} \\ &= \int_{\underline{x} + C(\epsilon)} \Phi(\underline{z} - \underline{x}) g(\underline{z}, \exp(it)) d\underline{z} \\ &= \int_{\mathbf{R}^m} \Phi(\underline{z} - \underline{x}) g(\underline{z}, \exp(it)) d\underline{z}. \end{aligned}$$

The function h is smooth on $U \times S^1$ as Φ is smooth with respect to $\underline{x}$ and g is smooth with respect to t. Thus the first requirement of the lemma is satisfied. Since $\tilde{f} = f(1 - \Psi) + h$ and Ψ and h are smooth, the class of $\tilde{f}$ on any open set is no less than the class of f. Hence the fourth requirement is satisfied.

Now

$$\tilde{f}(\underline{x}, \exp(it)) = f(\underline{x}, \exp(it)) + h(\underline{x}, \exp(it)) - g(\underline{x}, \exp(it)).$$

To satisfy the third requirement we need only choose ϵ small enough so that, on $U \times S^1$, h and g and all their derivatives up to order k are δ-close everywhere on fibres. By a mean value theorem, given $\underline{x}$ in U and $\exp(it)$ in S^1, there exists some $\underline{y}_0$ in $C(\epsilon)$ such that

$$h(\underline{x}, \exp(it)) = g(\underline{x} + \underline{y}_0, \exp(it))$$

and some $\underline{y}_l$ in $C(\epsilon)$ such that

$$\begin{aligned} \frac{d^l h_{\underline{x}}(t)}{dt^l} &= \frac{d^l \int_{\mathbb{R}^m} \Phi(\underline{z}-\underline{x}) g_{\underline{z}}(t) d\underline{z}}{dt^l} \\ &= \int_{\mathbb{R}^m} \Phi(\underline{z} - \underline{x}) \frac{d^l g_{\underline{z}}(t)}{dt^l} d\underline{z} \\ &= \frac{d^l g_{\underline{x}+\underline{y}_l}(t)}{dt^l}, \end{aligned}$$

where $h_{\underline{x}}(t) = h(\underline{x}, \exp(it))$ and $g_{\underline{x}}(t) = g(\underline{x}, \exp(it))$, $1 \leq l \leq k$. The functions g and $\frac{\partial^l g}{\partial t^l}$, $1 \leq l \leq k$, are uniformly continuous on the compact space $\overline{V} \times S^1$. Choose ϵ small enough that

$$|g(\underline{x}, \exp(it)) - g(\underline{x}^*, \exp(it))| < \delta$$

and

$$\left| \frac{d^l g_{\underline{x}}(t)}{dt^l} - \frac{d^l g_{\underline{x}^*}(t)}{dt^l} \right| < \delta$$

if $\underline{x}$ and $\underline{x}^*$ are in V and $|\underline{x} - \underline{x}^*| < \epsilon$. Then the third requirement is satisfied. □

Lemma 9.3.5 *Let $\pi : E \to B$ be a smooth compact fibre bundle with fibre S^1 and let $f : E \to \mathbf{R}$ be continuous and smooth on fibres. If $k \geq 0$ then there exists some smooth real-valued function on E which approximates f on fibres as closely as required in the Whitney C^k topology.*

Proof We construct a suitable smooth function inductively. Let the maps

$$h_i \ : \ U_i \times S^1 \ \to \ E$$

determine smooth local trivialisations of the bundle $\pi : E \to B$ such that

$$E = \cup_{i=1}^{q} h_i(A_i \times S^1),$$

the U_i are open sets in $\mathbf{R}^m$ and where A_i, V_i and W_i are chosen as in the previous smoothing lemma.

Let f_0 equal f. Assume that $f_{i-1} : E \to \mathbf{R}$ is a continuous function, smooth on fibres and smooth on

$$\bigcup_{j=1}^{i-1} h_j(W_j \times S^1)$$

for some i, $1 \le i \le q$. Let

$$g_{i-1} = f_{i-1} \circ h_i : U_i \times S^1 \to \mathbf{R}.$$

Choose $\delta > 0$ and apply the previous smoothing lemma to obtain a continuous function

$$g_i : U_i \times S^1 \to \mathbf{R}$$

such that

1. g_i is of class C^∞ on a neighbourhood of $A_i \times S^1$,
2. g_i equals g_{i-1} outside $V_i \times S^1$,
3. g_i is $\frac{\delta}{2^i}$ close to g_{i-1} everywhere,
4. the derivatives of g_i on fibres, up to order k, are $\frac{\delta}{2^i}$ close to those of g_{i-1} everywhere and
5. g_i is smooth on fibres.

Then f_i is well defined by the equations

1. $f_i = f_{i-1}$ outside $h_i(U_i \times S^1)$,

2. $f_i \circ h_i(\underline{x}, \exp(it)) = f_{i-1} \circ h_i(\underline{x}, \exp(it))$ for $\underline{x}$ outside V_i and

3. $f_i \circ h_i(\underline{x}, \exp(it)) = g_i(\underline{x}, \exp(it))$ for $\underline{x}$ in U_i.

The function f_i is smooth on $\bigcup_{j=1}^{i} h_j(W_j \times S^1)$. By induction, there exists a smooth function $f_q : E \to \mathbf{R}$ such that, f_q is δ-close to f and, on fibres, in local co-ordinates, the derivatives of f_q, up to order k, are δ-close to those of f. □

Corollary 9.3.6 *Let $\pi : E \to B$ be a smooth compact fibre bundle with fibre S^1. If $k \geq 2$ and there exists a continuous real-valued function on E which is in Ω_k on fibres then there exists a smooth real-valued function on E which is in Ω_k on fibres.*

Proof of Theorem 9.3.1. The existence of smooth Morse functions on the circle proves the case $n = 0$. The other cases are proved by induction on n. Assume that there exists a smooth function

$$f_{n-1} \ : \ S^{2n-1} \ \to \ \mathbf{R}$$

such that f_{n-1} is in Ω_{n+1} on fibres, for some $n \geq 1$. Consider the homeomorphism

$$\tilde{M}_n \ : \ S^1 \times D^{2n}/_{\sim} \ \to \ S^{2n+1}$$

defined in Lemma 9.3.3.

The preimages of fibres in S^{2n+1} are of the form

1. $S^1 \times \{\underline{z}\}$ where $|\underline{z}| < 1$ or

2. $\{[(1, \exp(i\omega)\underline{z})] : \exp(i\omega) \in S^1\}$ where $|\underline{z}| = 1$.

The homeomorphism $\tilde{M}_n$, when restricted to the preimage of any fibre, is a diffeomorphism onto its image. We use this property to construct a continuous function

$$\tilde{f} : S^{2n+1} \to \mathbf{R},$$

in Ω_{n+2} on fibres.

Let $i_{n+1} : \Omega_{n+1} \to \Omega_{n+2}$ be the inclusion defined in Corollary 9.2.4 and let $G : P \to Q$ be the map defined in §9.1. By Corollary 9.2.4, the map

$$i_{n+1} \circ G(f_{n-1}) \;:\; S^{2n-1} \;\to\; \Omega_{n+2}$$

induces the zero map on all homotopy groups and so extends continuously to

$$g_n : D^{2n} \to \Omega_{n+2}.$$

We construct a continuous function

$$h_n \;:\; S^1 \times D^{2n}/_\sim \;\to\; \mathbf{R}$$

such that h_n is in Ω_{n+2} on fibres, in two steps. Define

$$\begin{array}{rcll} h_n & : & S^1 \times (D^{2n})^0 \;\to\; \mathbf{R} & \text{by} \\ & & (\exp(i\omega), \underline{z}) \;\mapsto\; g_n(\underline{z})(\exp(i\omega)) & \end{array}$$

and

$$\begin{array}{rcll} h_n & : & S^1 \times S^{2n-1}/_\sim \;\to\; \mathbf{R} & \text{by} \\ & & [(\exp(i\omega), \underline{z})] \;\mapsto\; f_{n-1}(\exp(i\omega)\underline{z}). & \end{array}$$

Set $\tilde{f}_n$ equal to

$$h_n \circ \tilde{M_n}^{-1} \;:\; S^{2n+1} \;\to\; \mathbf{R}.$$

Then $\tilde{f}_n$ is continuous and in Ω_{n+2} on fibres. By Corollary 9.3.6, there exists some f_n in $C^\infty(S^{2n+1}, \mathbf{R})$, which is in Ω_{n+2} on fibres. By induction the theorem is true for every $n \geq 0$. □

9.4 Simple Singularities on Compact Principal S^1-Bundles

The main result of this chapter is the following theorem.

Theorem 9.4.1 *Let $\pi : E \to B$ be a smooth compact principal S^1-bundle and let m be the dimension of the base B, $2 \le m < \infty$. Then there exists a smooth function $f : E \to \mathbf{R}$ such that f is in $\Omega_{[\frac{m}{2}]+2}$ on fibres.*

Proof By Theorem A.4.1 there exists a bundle map

$$h \ : \ E \ \to \ S^{2n+1}$$

if $n \ge 1$ and $2 \le m \le (2n+1)$. Any bundle map is continuous and, in the case of principal S^1-bundles, a diffeomorphism when restricted to a fibre and its image. By Theorem 9.3.1 there exists a smooth function

$$f_n \ : \ S^{2n+1} \ \to \ \mathbf{R}$$

which is in Ω_{n+2} on fibres, for every $n \ge 0$. Hence, if $m = 2n$ or if $m = 2n+1$. then there exists a continuous function

$$f_n \circ h \ : \ E \ \to \ \mathbf{R}$$

which is in Ω_{n+2} on fibres. Hence, if $m = 2n$ or if $m = 2n+1$, then by Corollary 9.3.6 there exists a smooth function

$$f \ : \ E \ \to \ \mathbf{R}$$

which is in Ω_{n+2} on fibres. □

9.5 Regarding Functions as Sections

A smooth real-valued function on E which is in Ω_k on fibres may be regarded as a section of a bundle associated to $\pi : E \to B$. We shall not use these sections but construct the associated bundle for the sake of interest.

The space S^1 is a topological transformation group of the topological space Ω_k, $k \ge 2$, relative to the map

$$(\exp(i\omega), f) \mapsto \exp(i\omega) f \ : \ S^1 \times \Omega_k \ \to \ \Omega_k$$

where

$$(\exp(i\omega)f)(\exp(i\theta)) = f(\exp(i\omega)\exp(i\theta))$$

for every ω and θ in $\mathbf{R}/2\pi\mathbf{Z}$ and f in Ω_k (see [44, page 7]). Hence, S^1 is a topological transformation group of $E \times \Omega_k$ relative to the map

$$(\exp(i\omega), e, f) \mapsto (\exp(i\omega)e, \exp(i\omega)f) \quad : \quad S^1 \times E \times \Omega_k \quad \rightarrow \quad E \times \Omega_k$$

where $e \in E$. Let $E \times_{S^1} \Omega_k$ be the set of equivalence classes in $E \times \Omega_k$ defined by $(e, f) \sim (e', f')$ iff

$$e' = \exp(i\omega)e \text{ and } f' = \exp(i\omega)f$$

for some $\exp(i\omega)$ in S^1. Here $E \times_{S^1} \Omega_k$ is given the identification topology. Let $[(e, f)]$ denote the equivalence class of (e, f). Define

$$\begin{array}{rcl} \pi_{\Omega_k} \; : \; E \times_{S^1} \Omega_k & \rightarrow & B \qquad \text{by} \\ {[(e,f)]} & \mapsto & \pi(e). \end{array}$$

The space $E \times_{S^1} \Omega_k$ is the total space of a fibre bundle with projection π_{Ω_k}, base B, fibre Ω_k, group S^1 and the same system of co-ordinate transformations as its associated principal bundle $\pi : E \rightarrow B$ (see [44, page 43]).

Lemma 9.5.1 *Let $\pi : E \rightarrow B$ denote a smooth principal S^1-bundle with total space E and base space B of dimension m, $0 \leq m < \infty$. If, for some fixed $k \geq 2$,*

$$P = \{f \in C^0(E, \mathbf{R}) \text{ such that } f \in \Omega_k \text{ on fibres }\},$$

$$Q = \{g \in C^0(E, \Omega_k) \text{ such that } g \text{ is equivariant with respect to the action of } S^1\}$$

and R is the set of all continuous sections of

$$\pi_{\Omega_k} \; : \; E \times_{S^1} \Omega_k \; \rightarrow \; B$$

then P, Q and R are isomorphic. (For the purposes of this chapter we say two sets are isomorphic if there exists a map from one to the other which is both one-to-one and onto.)

Proof It is sufficient to find maps $G : P \to Q$, $S : Q \to R$ and $F : R \to P$ such that $F \circ S \circ G$ is the identity map on P, $G \circ F \circ S$ is the identity map on Q and $S \circ G \circ F$ is the identity map on R. As in §9.1, define

$$\begin{array}{rcl} G : P & \to & Q \\ f & \mapsto & G(f) \end{array} \quad \text{by}$$

where

$$\begin{array}{rcl} G(f) : E & \to & \Omega_k \\ e & \mapsto & \{\exp(i\omega) \mapsto f(\exp(i\omega)e)\}. \end{array} \quad \text{is defined by}$$

Note that $G(f)$ is equivariant with respect to the action of S^1.

Define

$$\begin{array}{rcl} S : Q & \to & R \\ g & \mapsto & S(g) \end{array} \quad \text{by}$$

where

$$\begin{array}{rcl} S(g) : B & \to & E \times_{S^1} \Omega_k \\ b & \mapsto & [e, g(e)] \end{array} \quad \text{is defined by}$$

and where $b \in B$ and $e \in F_b$.

Define

$$F : R \to P$$

by

$$\begin{array}{rcl} F(s) : E & \to & \mathbb{R} \\ e & \mapsto & f_e(\exp(i0)) \end{array} \quad \text{is}$$

where $s(\pi(e)) = [e, f_e]$. Clearly, $F \circ S \circ G$, $G \circ F \circ S$ and $S \circ G \circ F$ are the required identity maps. $\square$

Chapter 10

Constrained Critical Points

This chapter deals with the problem of determining the type of a critical point arising in the method of Lagrange multipliers. This method is the usual one used to find the critical points of a smooth function f defined on an n-manifold $M \subset \mathbf{R}^{n+m}$, a smooth submanifold given as the set where $g_i = c_i$ for smooth functions $g_i : \mathbf{R}^{n+m} \to \mathbf{R}$, $1 \leq i \leq m$.

The method consists of introducing a vector

$$\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$$

"of undetermined multipliers", defining L to be

$$f + \underline{\lambda}.\underline{g} = f + \Sigma_{i=1}^{m} \lambda_i g_i$$

and finding its critical points.

The question of deciding the nondegeneracy and type of a critical point is usually disregarded in the textbooks or dismissed as being too complicated. Our purpose is to show, on the contrary, that the criteria can be stated and derived in a straightforward manner.

We compare the Hessian of f restricted to M with the bordered Hessian, that is, the Hessian of L regarded as a function of $n + 2m$ variables (including $\underline{\lambda}$).

The two Hessians have the same nullity at corresponding critical points and when they are nondegenerate, they have the same signature.

10.1 Lagrange Multipliers and the Bordered Hessian

Let $U \subset \mathbf{R}^{n+m}$ be an open subset and

$$\underline{g} \ : \ U \ \rightarrow \ \mathbf{R}^m$$

be a C^1 function such that

$$d\underline{g}_{\underline{a}} \ : \ \mathbf{R}^{n+m} \ \rightarrow \ \mathbf{R}^m$$

has rank m for every $\underline{a}$ in M where

$$M = \{\underline{x} \in U \ : \ \underline{g}(\underline{x}) = \underline{c}\}.$$

Then, by the Implicit Function Theorem [12], M is a smooth n-dimensional manifold.

We wish to determine the critical points of the function

$$f_1 \ : \ M \ \rightarrow \ \mathbf{R}$$

which is the restriction of a C^2 function

$$f \ : \ U \ \rightarrow \ \mathbf{R}.$$

For $\underline{\lambda} \in \mathbf{R}^m$, we consider the Lagrangian

$$L = f + \underline{\lambda}.(\underline{g} - \underline{c})$$

either as a function of $\underline{x}$ in U or as a function of $(\underline{x}, \underline{\lambda})$ in $U \times \mathbf{R}^m$. The critical points are obtained by solving the equations

$$\nabla L = \underline{0} \ \text{ and } \ \underline{g} = \underline{c}$$

or, equivalently,

$$\nabla L = \underline{0}$$

regarding L as a function of $(\underline{x}, \underline{\lambda})$.

Let $H_M f(\underline{a})$ be the Hessian form of f_1 on $T_{\underline{a}}(M)$, the tangent space to M at the critical point $\underline{a}$. We call $H_M f(\underline{a})$ the restricted Hessian of f_1 at the critical point $\underline{a}$. If $H_M f(\underline{a})$ is nondegenerate then f_1 is a Morse function at $\underline{a}$ and the index of the critical point, that is, the number of independent directions in which f_1 decreases, is determined by the signature of the form $H_M f(\underline{a})$. We give a practical method for determining when $H_M f(\underline{a})$ is nondegenerate and for calculating its signature (see Definition 10.2.1).

Let $\underline{g}$ be a C^2 function and let $HL(\underline{a}, \underline{\lambda})$ be the bordered Hessian of L at the critical point $(\underline{a}, \underline{\lambda})$ of L; that is, the Hessian of L regarded as a bilinear form on $T_{(\underline{a},\underline{\lambda})}(U \times \mathbf{R}^m) \cong \mathbf{R}^{n+2m}$.

If

$$\underline{g}^T = (g_1, g_2, \ldots, g_m)$$

let

$$d\underline{g}^T = (\nabla g_1, \nabla g_2, \ldots, \nabla g_m)$$

denote the transpose of the Jacobian matrix of $\underline{g}$ at $\underline{a}$. Then the matrix of the bordered Hessian $HL(\underline{a}, \underline{\lambda})$ is

$$\begin{pmatrix} Hf + \underline{\lambda}.H\underline{g} & d\underline{g}^T \\ d\underline{g} & 0 \end{pmatrix}$$

where $\underline{\lambda}$ is evaluated by solving the equation

$$\underline{0} = \nabla L = \nabla f + \underline{\lambda}.d\underline{g}$$

at $\underline{a}$.

10.2 The Main Result

Definition 10.2.1 *Let a symmetric bilinear form on a real vector space be represented by the matrix H. Then the nullity of the symmetric bilinear form is the dimension of the kernel of H and if the nullity is zero, its signature is the difference between the number of positive eigenvalues of H and the number of negative ones (see [45, page 259]).*

Theorem 10.2.2 *1. The nullity of $H_M f(\underline{a})$ equals the nullity of $HL(\underline{a}, \underline{\lambda})$.*

2. If $HL(\underline{a}, \underline{\lambda})$ (and hence $H_M f(\underline{a})$) is nondegenerate, then the signature of $H_M f(\underline{a})$ equals the signature of $HL(\underline{a}, \underline{\lambda})$.

Theorem 10.2.2 follows from the purely algebraic Theorem 10.2.4, using Taylor's formula and the Implicit Function Theorem [12]. When it is applied to critical points as above, it yields the following result.

Corollary 10.2.3 *The point $\underline{a}$ in M is a critical point of $f_1 : M \to \mathbf{R}$ if and only if $(\underline{a}, \underline{\lambda})$ is a critical point of $L : U \times \mathbf{R}^m \to \mathbf{R}$. In this case, $\underline{a}$ is nondegenerate if and only if $(\underline{a}, \underline{\lambda})$ is nondegenerate; and the index $I(f_1, \underline{a})$ of f_1 at $\underline{a}$ is related to the index $I(L, \underline{a}, \underline{\lambda})$ of L at $(\underline{a}, \underline{\lambda})$ by*

$$I(f_1, \underline{a}) + m = I(L, \underline{a}, \underline{\lambda}).$$

So, for example, $\underline{a}$ is a local minimum of f_1 if $(\underline{a}, \underline{\lambda})$ is a nondegenerate critical point of L of index m. Similarly, $\underline{a}$ is a local maximum of f_1 if $(\underline{a}, \underline{\lambda})$ is a nondegenerate critical point of L of index $n + m$.

Theorem 10.2.4 *Let*

$$C = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

be the real symmetric matrix consisting of the $(n+m)\times(n+m)$ symmetric matrix A, the $(m)\times(n+m)$ matrix B of rank m and the zero $m\times m$ matrix 0. The symmetric bilinear form induced on $KerB$ by A is denoted by b. Then the bilinear form on $\mathbf{R}^{n+2m}$ defined by C is isomorphic to $b\oplus H$ where H is the $2m$-dimensional hyperbolic form

$$H = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Before proving Theorem 10.2.4 we include three standard facts from linear algebra we shall require.

Fact 10.2.5 *Let $L^T : \mathbf{R}^k \to \mathbf{R}^l$ be the transpose of the matrix $L : \mathbf{R}^l \to \mathbf{R}^k$, then*

$$(KerL)^\perp = ImL^T.$$

Proof Let r denote the rank of L. Then $\dim(\mathrm{Ker}L) = l - r$ and hence $\dim(\mathrm{Ker}L)^\perp = r$. Also $\dim(\mathrm{Im}L^T) = r$. It is therefore enough to show that $\mathrm{Im}L^T \subset (\mathrm{Ker}L)^\perp$, that is

$$\mathrm{Im}L^T \perp \mathrm{Ker}L.$$

Suppose $\underline{x} \in \mathbf{R}^k$ and $\underline{z} \in \mathrm{Ker}L$ then

$$\underline{z}^T(L^T\underline{x}) = \underline{x}^T(\mathrm{L}\underline{z}) = 0.$$

□

Definition 10.2.6 *Let b be a bilinear form on the real finite dimensional space V. The annihilator of a subspace $U \subset V$ is*

$$U^\perp = \{\underline{x} \in V \,:\, b(\underline{x},\underline{u}) = 0 \;\forall\, \underline{u} \in U\}.$$

Fact 10.2.7 *Let b be a nondegenerate symmetric bilinear form on V of dimension $2m$ and let $W \subset V$ have dimension m and $W \subset W^{\perp}$, then b is represented by the matrix*

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Proof Choose $\underline{w} \neq 0$ in W and $\underline{v}$ such that $b(\underline{w}, \underline{v}) = 1$. Let $\underline{u} = \underline{v} - b(\underline{v}, \underline{v})\underline{w}/2$, then $b(\underline{u}, \underline{u}) = 0$ and $\{\underline{u}, \underline{w}\}$ is the required basis in the case $m = 1$. When $m > 1$, let $U = \mathrm{Span}\{\underline{u}, \underline{w}\}$ and consider $U^{\perp}$; this contains a subspace that is self-annihilating and of dimension $m-1$. The result follows by induction. □

Fact 10.2.8 *If b is a symmetric bilinear form on an n-dimensional real inner product space then there is an orthonormal basis*

$$\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$$

such that

$$b(\underline{e}_i, \underline{e}_j) = \delta_{ij} a_i$$

for some a_i in $\mathbf{R}$, $1 \leq i,\ j \leq n$.

For the proof (and many other basic linear algebra results) see Lang [26].

Proof of Theorem 10.2.4. If W denotes the m-dimensional subspace

$$\left\{ \begin{pmatrix} \underline{0} \\ \underline{y} \end{pmatrix} : \underline{y} \in \mathbf{R}^m \right\} \subset \mathbf{R}^{n+2m},$$

then by Fact 10.2.5 applied to B, there is a canonical orthogonal decomposition

$$\mathbf{R}^{n+2m} \cong \mathrm{Ker}B \oplus \mathrm{Im}B^T \oplus W.$$

By Fact 10.2.8 we can choose an orthonormal basis

$$\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$$

for $\mathrm{Ker}B$ such that $b(\underline{e}_i, \underline{e}_i) = a_i$ and $b(\underline{e}_i, \underline{e}_j) = 0$ for $i \neq j$. Then for $1 \leq i \leq n$, $\underline{e}_i$ is a vector whose component in W is zero. Since $\underline{e}_i \in \mathrm{Ker}B$ we have $C\underline{e}_i \in W^{\perp}$ and hence

$$C\underline{e}_i = \underline{k}_i + B^T \underline{f}_i$$

where $\underline{k}_i \in \mathrm{Ker}B$ and $\underline{f}_i$ in $W \cong \mathbf{R}^m$ is unique because B^T is one-to-one. Moreover, $\underline{k}_i = a_i\underline{e}_i$ because

$$\underline{e}_j^T C\underline{e}_i = \delta_{ij}a_i.$$

Hence,

$$C\underline{e}_i = a_i\underline{e}_i + B^T\underline{f}_i = a_i\underline{e}_i + C\underline{f}_i.$$

Define

$$K = \mathrm{Span}\{\underline{e}_i - \underline{f}_i \,:\, 1 \leq i \leq n\}.$$

The following steps will prove Theorem 10.2.4.

Step 1

K and $\mathrm{Im}B^T \oplus W$ are orthogonal with respect to C.

Step 2

The form defined by C on K is isomorphic to b.

Step 3

The form defined by C on $\mathrm{Im}B^T \oplus W$ is hyperbolic.

Proof of Step 1

Since

$$C(\underline{e}_i - \underline{f}_i) = a_i\underline{e}_i \in \mathrm{Ker}B,$$

$1 \leq i \leq n$, we have that $C(K) \subset \mathrm{Ker}B$ and $\mathrm{Ker}B$ is orthogonal to $\mathrm{Im}B^T \oplus W$.

Proof of Step 2

Since

$$(\underline{e}_i - \underline{f}_i)^T C(\underline{e}_i - \underline{f}_i) = a_i \delta_{ij},$$

$1 \leq i,\ j \leq n$, we have that $C|K$ is isomorphic to b.

Proof of Step 3

By choosing a basis for the image of B^T, we can take the matrix of C to have the following form

$$\begin{pmatrix} A_1 & A_2^T & 0 \\ A_2 & A_3 & I_m \\ 0 & I_m & 0 \end{pmatrix}.$$

Hence the matrix of $C|\mathrm{Im}B^T \oplus W$ has the form

$$\begin{pmatrix} A_3 & I_m \\ I_m & 0 \end{pmatrix}$$

and so is equivalent to a hyperbolic form by Fact 10.2.7, since W is a self-annihilating space of dimension m. □

10.3 Comparison With Classical Criteria

In the literature there are criteria for deciding when a critical point is a local maximum or minimum. We show how two of these criteria are related to our result (see [14] and [19]).

Criterion 10.3.1 *Let*

$$C = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

be as in Theorem 10.2.4 and assume that the last $m \times m$ submatrix of B is nonsingular, then the form induced by A on $KerB$ is positive definite if the determinants Δ_i for $0 \leq i \leq n$ have sign $(-1)^m$ where $\Delta_i = \det C_i$ and C_i is obtained from C by deleting its first i rows and columns.

The proof is inductive and is based on the following lemma.

Lemma 10.3.2 *Let H be a nonsingular symmetric real matrix and H_1 be obtained from H by deleting one row and the corresponding column. If H_1 is also nonsingular and index H is the number of negative eigenvalues of H then*

$$\textit{index } H = \begin{cases} \textit{index } H_1 \textit{ or} \\ \textit{index } H_1 + 1 \end{cases}$$

depending on whether $\det H$ *and* $\det H_1$ *have the same or opposite sign.*

Proof Let M and M_1 be maximal negative definite subspaces for H and H_1 respectively. Recall that the dimension of a maximal negative definite subspace is unique. Clearly,

$$\dim M_1 \leq \dim M \leq \dim M_1 + 1.$$

Also,

$$\operatorname{sign} \det H = (-1)^{\dim M}$$

and

$$\operatorname{sign} \det H_1 = (-1)^{\dim M_1}.$$

Hence $\det H$ and $\det H_1$ have the same sign iff $\dim M = \dim M_1$ as required. □

Proof of Criterion 10.3.1. Write

$$C_n = \begin{pmatrix} A_n & B_n^T \\ B_n & 0 \end{pmatrix}.$$

Then $\Delta_n = \det C_n = (-1)^m (\det B_n)^2$, so the sign of $\Delta_n = (-1)^m$ since B_n is nonsingular. By Fact 10.2.7, C_n is hyperbolic and so has index m. By the lemma above, $C = C_0$ has index m. As C is nonsingular, by Corollary 10.2.3, the form induced by A on Ker$B$ is positive definite. □

Another criterion discovered in the 19^{th} century is the following (see [19] for historical references).

Criterion 10.3.3 *Let*

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

be as in Theorem 10.2.4. Then the form induced by A on KerB is positive definite iff the roots of

$$\det \begin{pmatrix} A - tI & B^T \\ B & 0 \end{pmatrix} = 0$$

are all positive.

Note that the above equation is of degree n. The stronger result that the roots of the above equation are the eigenvalues of the form A restricted to Ker$B$ with the same multiplicities is an immediate consequence of Theorem 10.2.4 applied to the matrix

$$\begin{pmatrix} A - tI & B^T \\ B & 0 \end{pmatrix}$$

when t is a root of the equation above.

Example 10.3.4 *To find the critical points of*

$$f(x, y, z) = x^3 + y^3 + z^3$$

on the surface $x^{-1} + y^{-1} + z^{-1} = 1$. This example is taken from [14, page 94].

Let

$$L = x^3 + y^3 + z^3 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$$

then

$$\tfrac{\partial L}{\partial x} = 3x^2 - \lambda x^{-2}$$

et cetera and the bordered Hessian is

$$\begin{pmatrix} 6x + 2\lambda x^{-3} & 0 & 0 & -x^{-2} \\ 0 & 6y + 2\lambda y^{-3} & 0 & -y^{-2} \\ 0 & 0 & 6z + 2\lambda z^{-3} & -z^{-2} \\ -x^{-2} & -y^{-2} & -z^{-2} & 0 \end{pmatrix}.$$

The critical points are given by

$$x^4 = y^4 = z^4 = \lambda/3$$

and

$$x^{-1} + y^{-1} + z^{-1} = 1.$$

These are $x = y = z = 3$, $\lambda = 243$; $x = y = 1$, $z = -1$, $\lambda = 3$ *and two other solutions symmetrical with the latter.*

In the first case the Hessian is

$$\begin{pmatrix} 36 & 0 & 0 & -9^{-1} \\ 0 & 36 & 0 & -9^{-1} \\ 0 & 0 & 36 & -9^{-1} \\ -9^{-1} & -9^{-1} & -9^{-1} & 0 \end{pmatrix}$$

which is nondegenerate and has signature two, so the critical point has index zero; that is, it is a nondegenerate minimum.

In the second case the Hessian is

$$\begin{pmatrix} 12 & 0 & 0 & -1 \\ 0 & 12 & 0 & -1 \\ 0 & 0 & -12 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

which is nondegenerate and has signature zero. So this critical point (and the other two symmetrical with it) has index one; that is, it is a saddle point.

Early references that discuss the general problem are listed in [19, Chapter VI]. More recent references for the problem are [8, 9, 10, 11, 14, 24, 27, 28, 29, 31, 40, 43]. The results in this chapter have been accepted for publication as part of a joint paper with Professor E. G. Rees [20].

Appendix A

Standard Theory and Notation

A.1 Differentiable Maps

Let U be an open set in $\mathbf{R}^n$, $n \geq 1$,

$$f \ : \ U \ \rightarrow \ \mathbf{R}$$

a differentiable map and $\underline{x}$ in U. Denote by $\frac{\partial f(\underline{x})}{\partial x_i}$ the partial derivative of f with respect to x_i at $\underline{x}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers. Denote by

$$\frac{\partial^{|\alpha|} f}{\partial \underline{x}^{\alpha}} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$$

a higher order mixed partial derivative where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If $\alpha = (0, \ldots, 0)$, then $\frac{\partial^{|\alpha|} f}{\partial x^{\alpha}} = f$.

The map

$$f \ : \ U \ \rightarrow \ \mathbf{R}$$

is k-times differentiable, or of class C^k, for $k \geq 1$, if

$$(\frac{\partial^{|\alpha|} f}{\partial \underline{x}^{\alpha}})(\underline{x})$$

exists and is continuous for every n-tuple of non-negative integers α such that $|\alpha| \leq k$, at each $\underline{x}$ in U. The map f is smooth or C^{∞} on U if f is C^k on U for every $k \geq 1$.

Consider an open subset $U \subset \mathbf{R}^n$ and a map

$$f : U \rightarrow \mathbf{R}^m,$$

$m \geq 1$. The map f is C^k on U, $k \geq 1$ if each component f_j is C^k on U, for every j, $1 \leq j \leq m$. If each component f_j is smooth, then we say f is smooth.

If $f : U \rightarrow \mathbf{R}^m$ is continuous then we say f is C^0 on U.

A.2 Germs

Let X and Y be topological spaces, x in X and

$$g : X \rightarrow Y$$

a continuous map. The germ of g at x in X is denoted

$$g : (U, x) \rightarrow V$$

or

$$g : (U, x) \rightarrow (V, g(x))$$

where U and V are open sets in X and Y respectively, such that $x \in U$ and $g(x) \in V$.

A.3 Smooth Fibre Bundles

For a comprehensive study of fibre bundles see [22, 44].

We denote a co-ordinate bundle $\mathcal{B}$ by

$$\{E, \pi, B, X, G\}$$

where E is the total space, B is the base space, π is the projection, X is the fibre and G is an effective topological transformation group of X called the group of the bundle. A group G is effective if for each g in G, $gx = x$ for all x in X, implies g is the identity element in G.

A smooth bundle $\mathcal{B} = \{E, \pi, B, X, G\}$ is a co-ordinate bundle such that E, B, X and G are smooth (that is C^∞) finite dimensional manifolds and the projection π, the co-ordinate functions and the co-ordinate transformations are all smooth.

A.4 The Hopf Map

Let $\mathbf{CP}^n$ denote the space of complex lines through the origin in $\mathbf{C}^{n+1}$, $n \geq 0$. Let

$$\pi^n \;:\; S^{2n+1} \;\rightarrow\; \mathbf{CP}^n$$

be the map which assigns to each $\underline{z}$ in S^{2n+1} the complex line through $\underline{z}$. For every $n \geq 0$, the Hopf bundle is the principal S^1-bundle with total space S^{2n+1}, base $\mathbf{CP}^n$, projection π^n and fibre S^1. For every $n \geq 0$, the map π^n is called the Hopf map. The following theorem is proved in [44, §19.2, §19.4].

Theorem A.4.1 *Let $\mathcal{B} = \{E, \pi, B, S^1, S^1\}$ be a compact principal S^1-bundle whose base has dimension m which is at least two. If $n \geq 1$ and $m \leq 2n + 1$ then there exists a bundle map*

$$h \;:\; E \;\rightarrow\; S^{2n+1}.$$

It follows that h is a diffeomorphism when restricted to a fibre and its image.

A.5 Critical Points

Let X and Y be C^k manifolds, $k \geq 1$. Denote the C^k maps from X into Y by $C^k(X, Y)$. Let g be in $C^1(X, Y)$. Denote by TX the tangent bundle of X and by T_xX the tangent space to X at x in X. Denote by

$$dg \;:\; TX \;\rightarrow\; TY$$

the derivative of g and by

$$dg_x \;:\; T_xX \;\rightarrow\; T_{g(x)}Y$$

the derivative of g at x in X.

Definition A.5.1

1. *The corank of dg_x equals* $\min\{\dim(X), \dim(Y)\}$ *minus the rank of dg_x.*

2. *A point x in X is a critical point of g iff the corank of $dg_x > 0$ iff g is neither an immersion nor a submersion at x.*

3. *A point y in Y is a critical value of g if $y = g(x)$ where x is a critical point of g.*

4. *If x in X is not a critical point, it is a regular point.*

5. *If y in Y is not a critical value, it is a regular value.*

6. *The map g has a singularity at x in X iff x is a critical point of g.*

7. *The set of critical points of g is denoted Σg.*

8. *If x in X is a critical point of g then g defines a singular germ at x.*

For further details see [30, 44].

A.6 Jets

Suppose X is a smooth n-manifold and Y is a smooth p-manifold, $1 \leq n,\ p < \infty$. Consider smooth maps f and $g : X \to Y$ such that $f(x) = g(x) = y$ in Y for some x in X.

Definition A.6.1 *The map f has first order contact with g at x if*

$$df_x = dg_x : T_xX \to T_yY.$$

Definition A.6.2 *The map f has k-th order contact with g at x if*

$$df : TX \to TY$$

has (k-1)-th order contact with dg at every point in T_xX, $k \geq 1$. This is written $f \sim_k g$ at x.

Notation A.6.3 *Let $J^k(X,Y)_{(x,y)}$ denote the set of equivalence classes under "$\sim_k$ at x" of maps $f : X \to Y$ such that $x \in X$, $y \in Y$ and $f(x) = y$.*

Notation A.6.4 *Let*

$$J^k(X,Y) = \coprod_{(x,y)\in X\times Y} J^k(X,Y)_{(x,y)}.$$

An element σ in $J^k(X,Y)$ is called a k-jet or k-jet of maps from X into Y.

Definition A.6.5 *If $\sigma \in J^k(X,Y)$ then there exist x in X and y in Y for which $\sigma \in J^k(X,Y)_{(x,y)}$. The point x is called the source of σ and y is called the target of σ. The map*

$$\alpha \ : \ J^k(X,Y) \ \to \ X$$

which sends each jet to its source is called the source map. The map

$$\beta \ : \ J^k(X,Y) \ \rightarrow \ Y$$

which sends each jet to its target is called the target map.

The following results are proved in [15, Chapter II, §2].

The jet $j^k f(x)$ is an invariant way of describing the Taylor expansion of f at x up to order k, $1 \leq k < \infty$. We often write jf for $j^\infty f$.

Notation A.6.6 *Denote by* $J^{k,0}_{n,p}$ *the set of k-jets of germs* : $(\mathbf{R}^n, \underline{0}) \rightarrow (\mathbf{R}^p, \underline{0})$, $1 \leq k \leq \infty$, $1 \leq n$, $p \leq \infty$.

If k is finite then $J^{k,0}_{n,p}$ is isomorphic to the real vector space of polynomial maps from $\mathbf{R}^n$ into $\mathbf{R}^p$ with constant zero and degree not exceeding k. We give this real vector space the usual topology and give $J^{k,0}_{n,p}$ the topology induced by this identification.

Let the diffeomorphisms

$$\Psi_j \ : \ \mathbf{R}^n \ \rightarrow \ U_j, \ \ j \in \mathcal{J}$$

and

$$\Psi'_r \ : \ \mathbf{R}^p \ \rightarrow \ W_r, \ \ r \in \mathcal{R}$$

determine co-ordinate patches on the smooth manifolds X and Y respectively. Let

$$\begin{array}{rcl} \tilde{\pi} \ : \ J^k(X,Y) & \rightarrow & X \times Y \\ \sigma & \mapsto & (\alpha(\sigma), \beta(\sigma)). \end{array} \qquad \text{be defined by}$$

Then there exists a canonical map

$$\Psi_{jr} \ : \ \mathbf{R}^n \times \mathbf{R}^p \times J^{k,0}_{n,p} \ \rightarrow \ \tilde{\pi}^{-1}(U_j \times W_r).$$

Lemma A.6.7 *The space $J^k(X,Y)$, $1 \leq k < \infty$, is a smooth manifold with a system of smooth co-ordinates given by*

$$\{\tilde{\pi}^{-1}(U_j \times W_r) \; : \; j \in \mathcal{J}, \, r \in \mathcal{R}\}$$

and

$$\{\Psi_{jr} \; : \; j \in \mathcal{J}, \, r \in \mathcal{R}\}.$$

Lemma A.6.8 *If the map $f \, : \, X \to Y$ is smooth then the map*

$$\begin{array}{rcll} j^k f \; : \; X & \to & J^k(X,Y) & \text{defined by} \\ x & \mapsto & j^k f(x) & \end{array}$$

is smooth, $1 \leq k < \infty$.

It is proved in §2.2 that if $1 \leq k < \infty$ then $J^k(X,Y)$ is the total space of a smooth fibre bundle with base space $X \times Y$, projection $\tilde{\pi}$ and fibre $J^{k,0}_{n,p}$.

A.7 Nakayama's Lemma

Lemma A.7.1 *Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{K}$ a field, $\mathcal{M}$ an ideal such that for x in $\mathcal{M}$, $(1+x)$ is invertible. Let C be a finitely generated $\mathcal{R}$-module and A a sub-module. Then*

1. *if $A + \mathcal{M}C = C$ then $A = C$.*

2. *if $\mathcal{R}$ is a $\mathcal{K}$-algebra and $\dim_{\mathcal{K}}(C/(A + \mathcal{M}^{d+1}C)) \leq d$ then $\mathcal{M}^d C \subseteq A$.*

For proof see [48, page 489].

A.8 Some Algebraic Geometry

All the facts discussed in this section may be found in [6]. Let $\mathbf{K}$ be a field and let $\mathbf{K}^n$ be the vector space of n-tuples over $\mathbf{K}$.

Definition A.8.1 *A subset $A \subset \mathbf{K}^n$ is called algebraic if there are polynomials $f_1, f_2, \ldots f_r$ in $\mathbf{K}[\underline{x}] = \mathbf{K}[x_1, x_2, \ldots x_n]$ such that*

$$A = \{\underline{x} \in \mathbf{K}^n \;:\; f_1(\underline{x}) = \ldots f_r(\underline{x}) = 0\}.$$

If $A \subset \mathbf{K}^n$ is an arbitrary set, then the ideal of all polynomials vanishing on A is denoted $\eta(A)$. Hence

$$\eta(A) = \{f \in \mathbf{K}[\underline{x}] \;:\; f(\underline{a}) = 0,\ \forall \underline{a} \in A\}.$$

Conversely, an ideal $\mathcal{A} \subset \mathbf{K}[\underline{x}]$ defines a subset $V(\mathcal{A}) \subset \mathbf{K}^n$. This is the set of zeroes of $\mathcal{A}$:

$$V(\mathcal{A}) = \{\underline{x} \in \mathbf{K}^n \;:\; f(\underline{x}) = 0,\ \forall f \in \mathcal{A}\}.$$

Theorem A.8.2 Hilbert's Basis Theorem

The polynomial ring $\mathbf{K}[\underline{x}]$ is Noetherian, that is, every ideal is finitely generated.

For proof see [26, page 144].

Corollary A.8.3 *The subset $V(\mathcal{A})$ is algebraic for any ideal $\mathcal{A} \in \mathbf{K}[\underline{x}]$.*

If $V(\eta(A)) = A$, then A is algebraic. For any subset A, $A \subseteq V(\eta(A))$. In general, $V(\eta(A))$ is the smallest algebraic set containing A.

Using Hilbert's Basis Theorem it can be proved that every strictly decreasing sequence of algebraic sets is finite. Hence we may give $\mathbf{K}^n$ a topology in which the algebraic sets are the closed sets. This is called the Zariski topology. It is much weaker than the usual one when $\mathbf{K}$ is $\mathbf{R}$ or $\mathbf{C}$.

Definition A.8.4 *An algebraic set A is called irreducible or a variety if whenever A_1 and A_2 are algebraic and $A = A_1 \cup A_2$ then $A = A_1$ or $A = A_2$.*

A variety cannot be decomposed into smaller algebraic sets. An arbitrary algebraic set A, which is not irreducible, can be decomposed into algebraic sets

$$A = A_1 \cup A_2,$$

such that $A \neq A_1$ and $A \neq A_2$. If this process is iterated we finally arrive at a decomposition

$$A = A_1 \cup A_2 \ldots \cup A_s$$

into irreducible sets. This decomposition is unique up to the order of its elements. The irreducible sets in the unique decomposition of A are called the irreducible components of A. From now on suppose $\mathbb{K}$ equals $\mathbb{R}$ or $\mathbb{C}$.

Definition A.8.5 *If $\mathcal{A} \subset \mathbb{K}[\underline{x}]$ is an ideal, then the rank of $\mathcal{A}$ is defined to be*

$$\rho(\mathcal{A}) = \max_{\underline{x} \in V(\mathcal{A})} Rk_{\underline{x}}(f_1, \ldots, f_k)$$

where $f_1, \ldots, f_k$ is any system of generators for $\mathcal{A}$, and

$$Rk_{\underline{x}}(f_1, \ldots, f_k) = Rank\left(\frac{\partial f_i(\underline{x})}{\partial x_j}\right),$$

$1 \leq i \leq k$, $1 \leq j \leq n$.

The rank ρ does not depend on the choice of generators.

The algebraic dimension of an algebraic set is defined in terms of transcendence degree. It turns out that if V is a variety, the dimension of V, denoted $\dim(V)$, is equal to the corank of $\eta(V)$, that is,

$$\dim(V) = n - \rho(\eta(V)).$$

The codimension of V, denoted $\mathrm{codim}(V)$, is equal to $\rho(\eta(V))$.

Lemma A.8.6 *If V and W are varieties and $V \subseteq W$, then $\dim(V) \leq \dim(W)$. Also, $\dim(V) = \dim(W)$ iff $V = W$.*

Definition A.8.7 *If A is an algebraic set defined by the polynomials $f_1, \ldots, f_k$, then the singular locus of A is given by*

$$\Sigma A = \{\underline{x} \in A \ : \ Rk_{\underline{x}}(f_1, \ldots, f_k) \text{ is not maximal on } A\}.$$

The set $A - \Sigma A$ is called the regular locus of A. Its points are called regular.

By definition of the rank of an ideal, the regular locus of A is not empty. Hence ΣA is strictly smaller than A. Moreover, ΣA is an algebraic set defined by the vanishing of all $(\rho \times \rho)$-minors in the matrix

$$\left(\frac{\partial f_i(\underline{x})}{\partial x_j} \right),$$

$1 \le i \le k, 1 \le j \le n$.

The connection between the algebraic and topological definitions of dimension for algebraic sets is clarified below.

Theorem A.8.8 *If V in K^n is a variety, the regular locus $V - \Sigma V$ is a real (or complex) analytic manifold of dimension $n - \rho(\eta(V))$. The singular locus ΣV is an algebraic set of dimension less than $n - \rho(\eta(V))$.*

Theorem A.8.9 *Let A be an algebraic set. Then A is the disjoint union of a finite number of analytic manifolds with dimensions less than or equal to the dimension of A. At least one manifold has the same dimension as A. The component analytic manifolds are unique up to order.*

If A is an algebraic set, the algebraic dimension of A is the maximum of the topological dimensions of the component analytic manifolds.

Lemma A.8.10 *The projection*

$$\begin{array}{rcrcl} \pi & : & \mathbb{K}^{n+1} & \rightarrow & \mathbb{K}^n \\ & & (x_1, \ldots, x_{n+1}) & \mapsto & (x_1, \ldots, x_n) \end{array}$$

is open in the Zariski topology.

Theorem A.8.11 *Let V in $\mathbb{K}^n$ be irreducible, then $\pi^{-1}(V) \subset \mathbb{K}^{n+1}$ is also irreducible and codim$(V) =$ codim$(\pi^{-1}(V))$.*

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